Calhoun: The NPS Institutional Archive

DSpace Repository

Theses and Dissertations 1. Thesis and Dissertation Collection, all items

1968-06

# Optimal Inventory Operating Policies Under Relaxed Delivery Assumptions.

Cook, John Morse

Monterey, California. Naval Postgraduate School

http://hdl.handle.net/10945/40054

This publication is a work of the U.S. Government as defined in Title 17, United States Code, Section 101. Copyright protection is not available for this work in the United States.

*Downloaded from NPS Archive: Calhoun*

Calhoun is the Naval Postgraduate School's public access digital repository for research materials and institutional publications created by the NPS community. Calhoun is named for Professor of Mathematics Guy K. Calhoun, NPS's first appointed -- and published -- scholarly author.

Dudley Knox Library / Naval Postgraduate School
411 Dyer Road / 1 University Circle
Monterey, California USA 93943

http://www.nps.edu/library

# UNITED STATES
# NAVAL POSTGRADUATE SCHOOL

# THESIS

OPTIMAL INVENTORY OPERATING POLICIES
UNDER RELAXED DELIVERY ASSUMPTIONS

by

John Morse Cook

June 1968

THESIS
C74724

[redacted]

[redacted]

# OPTIMAL INVENTORY OPERATING POLICIES UNDER RELAXED DELIVERY ASSUMPTIONS

by

John Morse Cook
Lieutenant, Supply Corps, United States Navy
A.B., Princeton University, 1959

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
June 1968

Signature of Author John Morse Cook

Approved by David A. Schrady
Thesis Advisor

Jack R Borsting
Chairman, Department of Operations Analysis

W. F. Koehler for R. F. Rinehart
Academic Dean

Thesis
C74724
c.2

[redacted]

# ABSTRACT

Historically, the inventory control problem has been analyzed under the assumption that the exact quantity ordered was delivered in one lot, at the end of a probabilistic or deterministic lead time, or, alternatively, that receipts are the direct result of a continuous, fixed rate production process. In this paper, the effect of relaxation of these assumptions is considered. It is quite common in large inventory systems to have the total quantity ordered delivered in increments phased over time. Contracts providing for delivery of short or excess quantities within tolerances are also frequently encountered. Production rates may be stochastic due to work stoppages, the diversion of resources to meet higher priority requirements and other external factors. A family of models, with a variety of delivery conditions, are presented. The optimal operating policies and costs of these models are compared to the policies and costs resulting from models with conventional delivery assumptions.

DUDLEY KNOX LIBRARY
NAVAL POSTGRADUATE SCHOOL
MONTEREY CA 93943-5101

TABLE OF CONTENTS

Chapter Page

1. Introduction 7

1.1 Background 7

1.2 Methodology and Assumptions 8

1.3 Conventional Delivery Models 12

2. Introduction to Phased Delivery Model Structure 18

2.1 General Discussion and Definitions 18

2.2 A Simple Theoretical Model 21

2.3 Comparison with Finite Production Rate Model 27

3. Constrained, Phased Delivery Models with Equal Delivery Lot Size at Equal Intervals 28

3.1 General Discussion 28

3.2 Delivery Size and Interval Fixed 31

3.3 Delivery Interval Fixed 37

3.4 Full Control of Delivery Size and Interval 45

4. Constrained Phased Delivery Models with Unequal Delivery Lots and Unequal Intervals 48

4.1 General Discussion 48

4.2 Fractional Delivery at Fixed Unequal Intervals 49

4.3 Increasing Delivery Lot Size at Equal Intervals 66

4.4 Equal Delivery Lots at Decreasing Intervals 71

5. Special Considerations in Phased Delivery Models 74

5.1 Shortages 74

5.2 Stochastic Parameters 88

5.3 Reducible Models 93

Chapter Page

6. Stochastic Delivery Quantities 95

7. Other Relaxations of the Conventional Delivery Assumptions 103

 7.1 Stochastic and Variable Production Rates 103

 7.2 Variable Lead Times and Expediting 110

 7.3 Multiple Sources with Different Delivery Patterns 114

8. Summary and Conclusions 117

BIBLIOGRAPHY 121

# LIST OF FIGURES

| Figure | | Page |
|---|---|---|
| 1. | Wilson "Q" Model | 14 |
| 2. | Finite Production Model | 16 |
| 3. | N- Increment Delivery Model | 23 |
| 4. | Delivery Size and Interval Fixed | 32 |
| 5. | Contour Map of K over n,q Plane | 40 |
| 6. | Typical "Valley" and Feasible Solution Set | 42 |
| 7A. | Two Increment Model: $q_1 = f_1Q > \lambda\tau$ | 50 |
| 7B. | Two Increment Model: $q_1 = f_1Q = \lambda\tau$ | 51 |
| 7C. | Two Increment Model: $q_1 = f_1Q < \lambda\tau$ | 52 |
| 8. | Symmetry of Two Increment Model | 56 |
| 9A. | Two Increment Model: $T_w < \tau < 2T_w$ | 57 |
| 9B. | Two Increment Model: $2T_w < \tau < 3T_w$ | 58 |
| 10A. | Two Increment Model with Shortages at End of Cycle Only | 61 |
| 10B. | Two Increment Model with Shortages Before Each Delivery | 61 |
| 11. | Multi-Increment Model | 64 |
| 12. | Increasing Production Rate: Delivery at Fixed Intervals | 67 |
| 13. | Increasing Production Rate: Fixed Delivery Quantities | 72 |
| 14. | Incremental Deliveries With Shortage Levels | 75 |
| 15. | Cost Function for K(i) | 77 |
| 16. | Shortage Model: Approximation One | 79 |
| 17. | Shortage Model: Approximation Two | 83 |
| 18. | Shortage Model: Approximation Three | 85 |

Figure Page

19. Approximate Shortage Model 86

20. Stochastic Demand and Production 105

21. Single Period, Variable $\Psi$ Model 108

22. Optimal Lead Time Model 113

# CHAPTER 1

# INTRODUCTION

## 1.1 Background

The inventory control problem is as old as history itself, but the first analytic studies of significance were those of Ford Harris and R. H. Wilson about 1915. These simple lot size models determine an optimal inventory policy in terms of a single variable, "The Economic Order Quantity." The assumptions required for this model are quite strict, and yet this simple model is useful in many applications. It is an excellent model for uncomplicated inventory systems where there is little motivation for assuming stochastic variation in demand. Efforts between world wars extended this simple deterministic model but it was not until the growth of the Management Sciences and Operations Research after World War II that attention was given to the stochastic nature of the problem and to the complex nature of many inventory systems. Recently multi-echelon systems have been examined and stochastic demand and lead time models have been analyzed. Consideration has been given to the special management problems of repairable item systems and other complicated systems. It is perhaps surprising, then, that virtually all of the literature deals with models for which very strict delivery assumptions are made.

It is almost universally assumed that either the exact quantity ordered arrives at a single instant at the end of a (stochastic or deterministic) lead time, or, alternatively, that the exact quantity ordered becomes available at a continuous fixed rate, starting at the end of a (stochastic or deterministic) lead time. Note that the first assumption is a special case of the second, with the delivery rate infinite. It has also been generally assumed that an order is placed with a single source, selected for offering the lowest unit price or by some other procedure outside of the model. Little attention has been given to determining an optimal operating policy under other delivery assumptions. Existing literature which does apply to this problem will be referenced when appropriate.

## 1.2 Methodology and Assumptions

The conventional delivery assumption actually consists of several ideas. First, the quantity is assumed to be exactly identical to the quantity ordered. Second, the receipt rate is assumed to be known, constant and continuous with the infinite rate being the most usual version. Third, delivery is assumed to be from one source, generally chosen outside of the model. Relaxation of each aspect of this assumption will be discussed, although this paper does not presume to remove all delivery restrictions or even to totally remove any one aspect of the restriction.

The basic approach will be to consider models which have relatively strict assumptions concerning other features of the system. This approach is necessary; otherwise the models would be so complex as to make the effects of the revised delivery assumptions very difficult to separate from other unrelated effects. In general, the models presented are either exact deterministic models or approximate probabilistic models. The occasional approximate deterministic model will be clearly labeled as to the approximations used and the reasons for doing so. In some cases, the probabilistic models are so designed as to provide qualitative information only, and some very rough heuristic rules are given as the only quantitative results. In some areas this type of result is very useful and exact models would have to assume knowledge of distributions and parameters which are not usually known.

The basic technique will be to determine optimal values for the decision variables, where the optimal values are taken to be those which minimize the average variable cost. This is equivalent to minimizing the average total cost. The phrase, decision variable, is used here to describe those parameters over which the inventory manager has control. The setting of values for these parameters is the way in which an inventory manager exerts his influence over the inventory system. A specific set of values for these parameters, in terms of the fixed inputs, constitutes an operating policy. The average costs, referred to above, are time averages over a sufficient period of time to cover several order cycles. In this paper, except where specifically discussed, it will always be assumed that the fixed input parameters will remain constant over a sufficiently long period of time to allow this assumption: time averages are an accurate reflection of the costs to be minimized.

Throughout this paper, decision variables will often be considered as if they were continuous when, in fact, they obviously take on only discrete integer values. The models could, of course, be formulated for integer solution, using difference equations. However, continuous models are generally used because it is easier to determine optimal values using the calculus. It is pointed out that the general problem addressed in this paper makes little sense unless the variables have fairly large values. Unless the order quantity and the demand rate are fairly large numbers, there is little reason to consider any delivery pattern but the usual one. The systems for which this paper's results are most significant and applicable will be ones for which demand is hundreds or thousands per year and order quantities are of comparable size.

Lead times are not specifically considered for the deterministic models. While this is not an assumption in the conventional sense, it is more conveniently considered at this point. The lead time, $\tau$, is defined as the time lapse between the placement of an order and the receipt of the first increment of material. In deterministic demand and lead time models, knowing the point in time at which the first receipt is required, it is easy to determine the time at which an order should be placed. Equivalently, knowing the amount of stock on hand required at the time of first receipt allows computation of a reorder level in terms of either inventory position (IP), or net inventory (NI). The basic factors which allow the lead times to be assumed equal to zero in these models are: that IP and NI differ only be the on order (00) quantity which has a time average of $\lambda\tau$, the lead time demand, and that each cycle will be identical to every other cycle. It is then possible to determine $\tau'$, such that $\tau' = \tau - mT$, and $mT \leq \tau < (m+1)T$, where m is an integer. Now $\tau'$ lies between zero and T. We can then compute a reorder time or reorder point using $\tau'$. When either demand or lead time is stochastic, detailed consideration is required within the model.

One assumption is not usually considered in the literature, but this author's experience with military inventory management prompts its inclusion. The fiscal climate must be assumed adequate for optimal or at least constrained optimal operation. Whether external constraints such as warehouse space, total inventory value or number of contracts per year are imposed or not, a minimal cost will be derived, subject to the conditions imposed. It must be assumed that the controller of the purse strings will provide the money necessary to cover these costs or, when appropriate, will close down the system. This assumption is basically equivalent to assuming that the system is operated as a commercial enterprize or, at least, is using profit maximization or cost minimization as the basic goal.

It is fairly obvious that inventory systems are not necessarily orientated primarily toward fiscal profit and loss. Military inventory systems are, of necessity, primarily oriented toward effective support of the operating forces in war or peace, with cost minimization as a secondary goal. A much broader approach is required to evaluate optimal policy in a trade off between maximal support (supply availability) and minimal cost, since the former is not easily translated into dollar terms. It is also obvious that certain items, because of their extremely critical nature, will be held at high inventory levels, regardless of the economic cost in the narrow sense. The climate in which the models of this paper operate will be assumed to be subject to constraint only in such limited senses as floor space or on hand value. A constraint on dollars spent will not be allowed, although such constraints exist in reality.

## 1.3 Conventional Delivery Models

For the convenience of the reader, the following results from inventory theory are given since they will be referenced often. They are the "simple lot size" or "Wilson Q" model and a simple deterministic finite production rate model.

The Wilson economic order quantity model minimizes average total cost for a system where no shortages are allowed and the entire quantity ordered is delivered at a single point in time. Demand is deterministic. It is assumed that the reader is familiar with this basic model and the purpose of presentation is to provide easy reference in the notation to be used throughout this paper and to introduce the notation in a familiar context. Figure 1 is a graphic representation of the model. Since the on order quantity is constant, it need not be considered in the cost minimization. The only decision variable is $Q$.

Let $K^{TC}$ = Total cost per cycle, where a cycle is the time period between successive orders or equivalently between successive deliveries.

$K$ = Variable cost per unit time.

$Q$ = Order quantity.

$C$ = Unit price.

$A$ = Order cost.

$\lambda$ = Demand rate (deterministic).

$I$ = Holding cost rate (applies to on hand only).

$b$ = Buffer stock (on hand at time of initial receipt).

$*$ = As superscript implies an optimal value.

The total cost is the sum of order cost, holding cost and the cost of the items. Now

$$K^{TC} = A + IC\left(\frac{Q}{2} + b\right)T + QC \ .$$

It is obvious, by inspection, that the cost will be minimal when $b = 0$, since $b < 0$ would imply shortages exist which is prohibited. To find $K$ it is necessary to divide $K^{TC}$ by $T = Q/\lambda$ and drop the fixed and zero cost terms. The variable cost per unit time is

$$K = \frac{A\lambda}{Q} + \frac{ICQ}{2} \ .$$

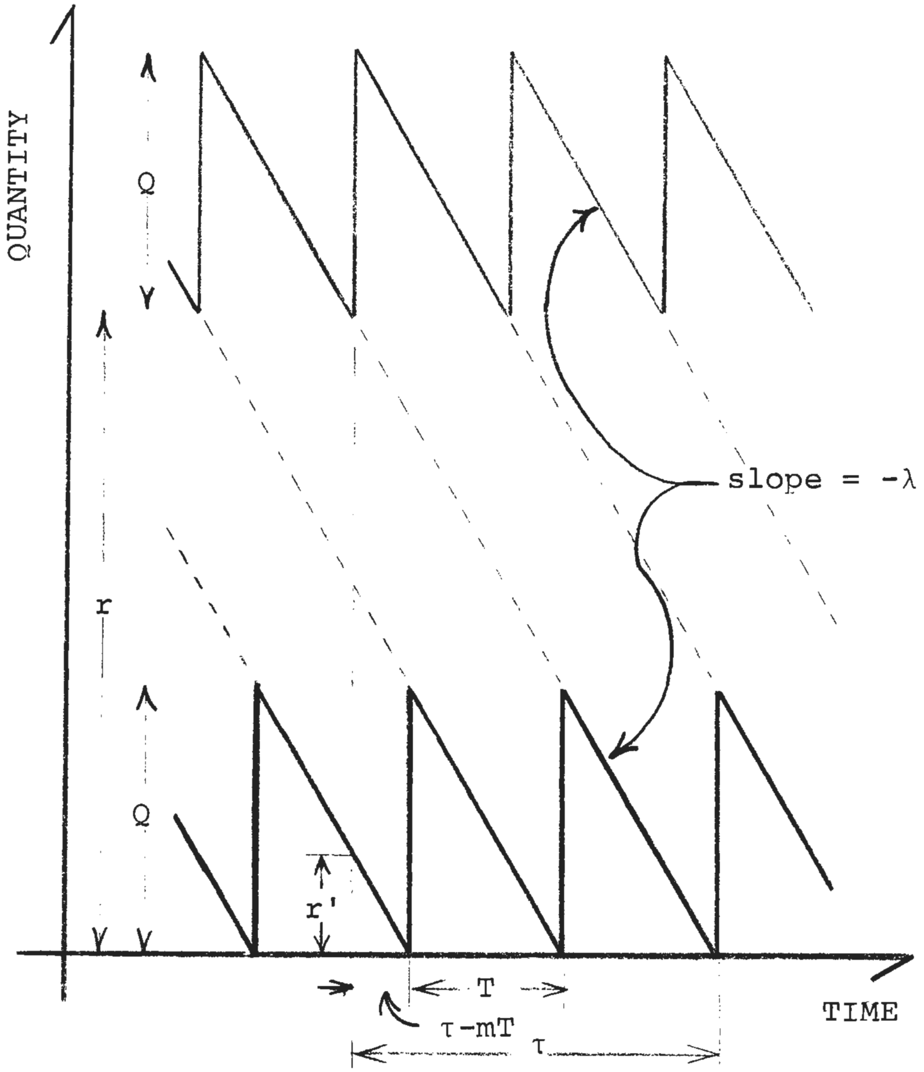


FIGURE 1

WILSON "Q" MODEL

To find the optimal value of the decision variable Q, set the derivative of K with respect to Q equal to zero and solve for Q.

$$\frac{dK}{dQ} = \frac{-A\lambda}{Q^2} + \frac{IC}{2} = 0$$

implies

$$Q = \sqrt{\frac{2A\lambda}{IC}} \; .$$

$$\frac{d^2K}{dQ^2} = \frac{2A\lambda}{Q^3} > 0$$

implies this value of Q minimizes K. Now the optimal values are seen to be

$$Q^* \equiv Q_W = \sqrt{\frac{2A\lambda}{IC}} \; ,$$

$$b^* = 0 \; ,$$

and

$$K^* \equiv K_W = \sqrt{2A\lambda IC} \; .$$

The simple deterministic finite production rate model varies from the Wilson Q model in that delivery is assumed to be the result of a continuous production process which allows immediate placement into stock. Figure 2 illustrates this model. Inventory level builds up at a rate of $\Psi - \lambda$ during the production portion of the cycle and then is depleted at the rate $\lambda$ during the remainder of the cycle. Two new symbols must be defined. Let $\Psi$ = continuous production rate as defined earlier and P = length of time during which production takes place. Now

$$K^{TC} = QC + A + ICT\left[\frac{Q}{2}\left(\frac{\Psi-\lambda}{\Psi}\right) + b\right] .$$

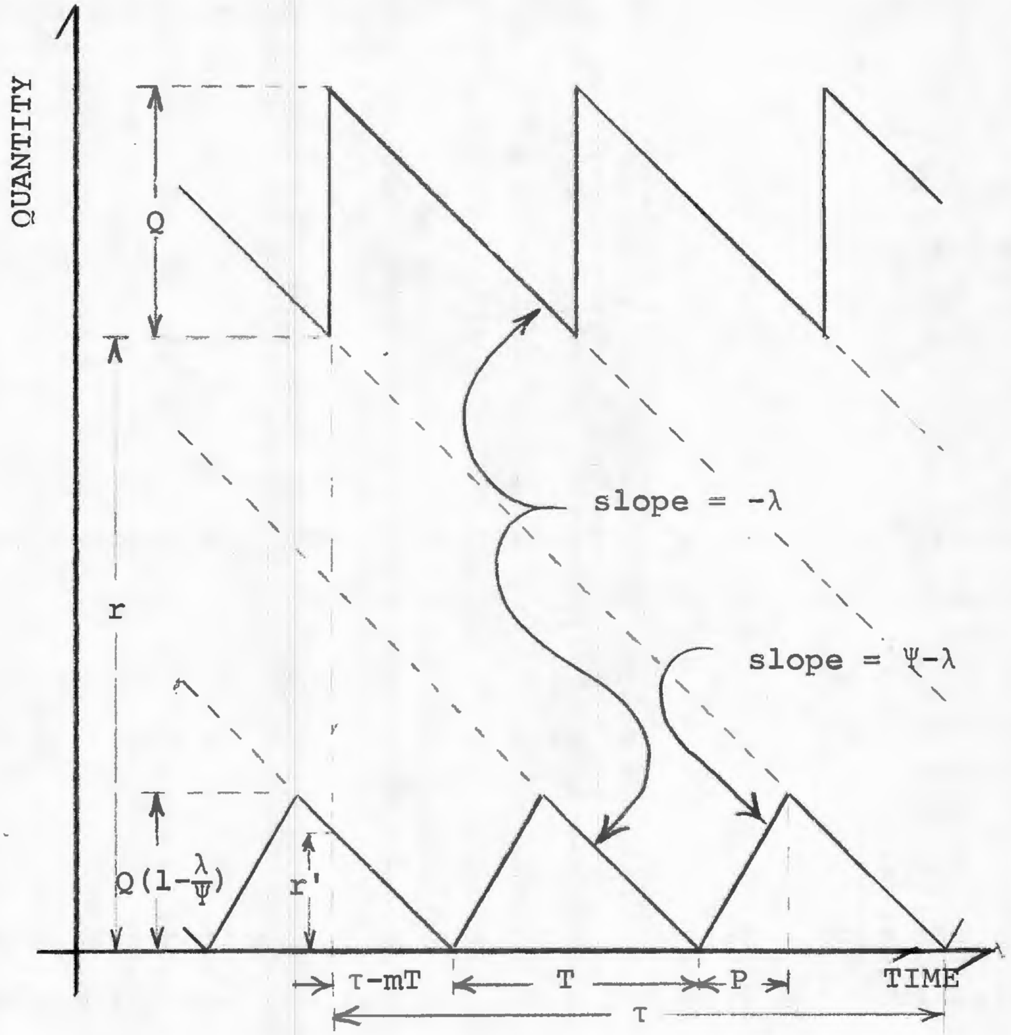


FIGURE 2

FINITE PRODUCTION MODEL

Again, it is obvious that $b^* = 0$. We divide by $T = Q/\lambda$ and drop the fixed and zero cost terms to find the variable cost per unit time,

$$K = \frac{A\lambda}{Q} + \frac{ICQ}{2}\left(\frac{\Psi-\lambda}{\Psi}\right) .$$

Set the derivative with respect to Q equal to zero and solve for Q. The second derivative with respect to Q is always greater than or equal to zero, so this value of Q yields a minimum K. The optimal values are

$$Q^* = Q_p = \sqrt{\frac{2A\lambda}{IC}\left(\frac{\Psi}{\Psi-\lambda}\right)} = Q_w \left(\frac{\Psi}{\Psi-\lambda}\right)^{\frac{1}{2}} ,$$

$$b^* = 0 ,$$

$$P^* = Q^*/\Psi ,$$

and

$$K^* \equiv K_p = K_w \left(\frac{\Psi}{\Psi-\lambda}\right)^{\frac{1}{2}} .$$

# CHAPTER 2

# INTRODUCTION TO PHASED DELIVERY MODEL STRUCTURE

## 2.1 General Discussion and Definitions

In chapters two through five, several models are presented. The assumption of a continuous finite or infinite delivery rate is relaxed, to allow incremental deliveries phased over a period of time. It is assumed that periodically, an incremental quantity $q_i$ will be received and after an interval $\tau_i$ another quantity $q_{i+1}$ will be received. The leadtime is $\tau_0$. Recall the lead time was previously defined to be the time between placement of an order and the receipt of the first delivery of units. The total quantity ordered, Q, will be assumed to be delivered in n lots of size $q_i$ such that

$$\sum_{i=1}^{n} q_i = Q \ .$$

The jth lot arrives at a time

$$\sum_{i=0}^{j-1} \tau_i$$

after the order is placed. The delivery is phased over a cycle time of

$$T = \sum_{i=0}^{n} \tau_i \ ,$$

where $\tau_n$ is the interval between the delivery of the nth (last) lot $q_n$ and the placement of the next order.

We will first consider models for which the delivery lot size and time intervals between them are equal for all values of i. That is $q_i = q$ for $i = 1,2,...,n$ and $\tau_j = \tau$ for

$j=1,2,\ldots,n-1$. We will then consider models in which the lot sizes and intervals may be arbitrary.

One common assumption is made for all of the following phased delivery models. When the number of deliveries is a (potential) variable, the order cost is of the form A + nB where A is order cost occurring as a result of placing the order and B is the component of order cost resulting from the delivery of an increment or lot. To a very close approximation the elements of order cost over a period of time are either proportional to the number of orders placed and may be included in A, or are proportional to the number of shipments received and may be included in B, or are proportional to the number of units purchased and may be included in the unit cost C of the item. For example, the administrative cost of letting a contract and set-up costs are included in A, the administrative cost of shipping and receiving a lot is included in B and the carrier's transportation charge per pound is included in C.

Inventory systems often receive ordered quantities in increments and could do so more frequently if convinced it was desirable. Many inventory managers have recognized the value of placing a contract against which calls may be made, which is a type of phased delivery. In fact, on 9 May 1967, Cyrus Vance, then Deputy Secretary of the Department of Defense, in a memorandum to the Secretary of the Army stated that deliveries from procurement should be scheduled into stock in proportion to anticipated issues, since, in this way, operating levels may be controlled without effecting the total order quantity or frequency of reordering.

A manufacturer may have some or all of the quantity of the item ordered on hand in his own inventory system, or he may have to produce the item after receiving the order. There is, in general, an administrative lead time and potentially (unless the entire quantity is available in the producer's inventory) a production lead time until the first item (except for a partial quantity on hand) comes off the end of an assembly line completed. All items may be completed as a batch, but more frequently, assembly line production will imply a finite discrete or continuous production rate. We will define a discrete delivery rate R which is dependent upon the production rate $\Psi$. However, R need not be equal to $\Psi$ and may be either larger or smaller than $\Psi$. Usually, in current practice, shipment is made in one batch after the entire quantity has been produced leading to $R = \infty$, the case of a single delivery which is covered well in the literature.

The assumption that the delivery rate is continuous and finite has also been well covered, but this assumption is unrealistic unless deliveries occur at very short time intervals (one a day perhaps). Charges for shipment must be insignificant or dependent only on the total units shipped. A combination warehouse factory or a warehouse adjoining a factory meets these criteria and the finite rate model is probably excellent for use in managing a producer's inventory of finished products located in a factory warehouse. However, this assumption rapidly loses its applicability, as distance enters the picture, or whenever two or more distinct activities are involved. If the source is some distance from the inventory location, it will not usually be feasible to ship continuously and we must consider R, the discrete analogue of $\Psi$. This paper will, hereafter, reserve the notation R for a discrete delivery rate and use $\Psi$ when a continuous production and delivery procedure is considered or as a limiting value for R. Basically, the reader is expected to think of R as fixed size shipments arriving at fixed time intervals. For example, R = 200 per month implies one shipment of 200 units is received once each month.

In addition, phased deliveries will usually decrease lead time which is almost universally accepted as a desired objective in "real world" systems. Obviously, it will be possible in many cases to have material shipped as it is produced rather than waiting for all units to be ready for delivery. This has the advantage of effectively reducing the lead time for a portion of the order quantity.

The models which follow represent a selection of theoretical and practical systems. The first is primarily designed as a starting point for analysis and to show the relationship between phased delivery and the conventional delivery assumption. The later models are designed to represent real systems and give practical guidance for setting operating policy under the conditions assumed.

## 2.2 A Simple Theoretical Model

This model is a no shortage model which assumes the manager has the option of getting some number, n, of equal sized shipments, q, at equal intervals, τ. The effect of the manager's control over each of the parameters (number, size and interval) is investigated.

The decision variables are n, q and τ. Shipment will be made in n equal sized lots, each of q units, at intervals τ. In the conventional delivery models $n = 1$, $q = Q$ and τ is undefined. Note that $\lambda\tau > q$ is infeasible since the demand rate λ exceeds the supply rate $R = q/\tau$. Therefore it will be required that $q \geq \lambda\tau$. Then $Q = qn$ and $T = qn/\lambda$. The order cost per cycle is $A + Bn$, so the order cost per unit time is

$$\frac{(A+Bn)\lambda}{qn} = \frac{A\lambda}{qn} + \frac{B\lambda}{q} .$$

Holding costs per cycle are proportional to the area under the net inventory curve. Reference to Figure 3 shows that an easy way to get this area is by subtracting the small parallelograms from the larger triangle with area $\frac{1}{2}\,QT$. Each parallelogram has area τq and there are

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$$

of them. Thus holding cost per cycle is

$$IC\left[\frac{QT}{2} - q\tau \sum_{i=1}^{n-1} i\right] = \frac{IC}{2}(QT - q\tau n(n-1))$$

$$= \frac{IC}{2}\,(qnT - T\lambda\tau(n-1)),$$

and holding cost per unit time is

$$\frac{IC}{2}\,(qn - \lambda\tau(n-1)).$$

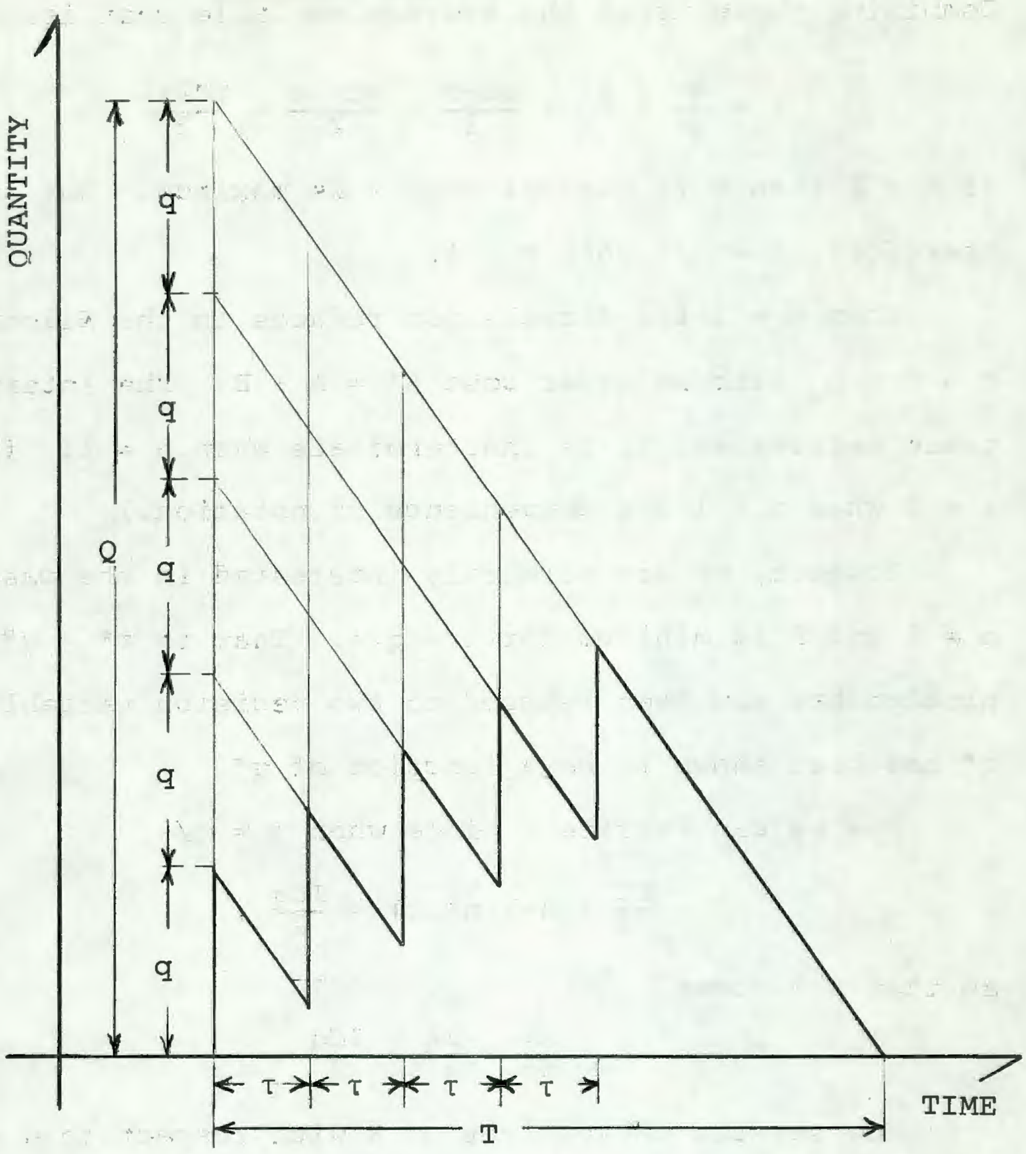


FIGURE 3

N-INCREMENT DELIVERY MODEL
(N=5)

Combining these terms the average variable cost is:

$$K = \frac{A\lambda}{qn} + \frac{B\lambda}{q} + \frac{ICqn}{2} - \frac{ICn\lambda\tau}{2} + \frac{IC\lambda\tau}{2} \, .$$

If $n \geq 1$ then K is minimal when $\tau$ is maximum. But $\lambda\tau \leq q$, therefore, $\tau = q/\lambda$ when $n > 1$.

When $n = 1$ the formulation reduces to the Wilson Q model, $q = Q = Q_W$ with an order cost $A' = A + B$. The interval between deliveries, $\tau$, is indeterminate when $n = 1$. (Define $\tau = 0$ when $n = 1$ for convenience of notation.)

However, we are primarily interested in the case when $n \neq 1$ and K is minimum for $\tau = q/\lambda$. That is $\tau^* = q^*/\lambda$. The problem has now been reduced to two decision variables since $\tau^*$ has been shown to be a function of $q^*$.

Now we can rewrite K since when $\tau = q/\lambda$,

$$\frac{IC}{2} (qn - \lambda\tau n + \lambda\tau) = \frac{ICq}{2} \, ,$$

so that K becomes

$$K = \frac{A\lambda}{nq} + \frac{B\lambda}{q} + \frac{ICq}{2} \, .$$

Now set the derivatives of K with respect to q and n equal to zero and solve.

$$\frac{\partial K}{\partial q} = -\frac{A\lambda}{nq^2} - \frac{B\lambda}{q^2} + \frac{IC}{2} = 0 \, ,$$

so that

$$q = \left[\frac{2(A+nB)}{nIC}\right]^{\frac{1}{2}} \, . \qquad (2.1)$$

$$\frac{\partial K}{\partial n} = \frac{-A\lambda}{n^2 q} = 0 \, . \qquad (2.2)$$

The Hessian is positive definite so we have a minimum. Now equation (2.2) implies that either $n^* = \infty$ and $q^* > 0$ or else

$q^* = \infty$ and $n^* > 0$. But n cannot be less than 1 so from equation (2.1)

$$q^2 = \frac{2A\lambda}{nIC} + \frac{2B\lambda}{IC} < \frac{2\lambda}{IC}(A+B) .$$

Therefore q is finite and the optimum must be with $n^* = \infty$. Now $n^* = \infty$ implies

$$q^* = \left[\frac{2B\lambda}{IC}\right]^{\frac{1}{2}} .$$

The optimal value of K is found by substituting $n^*$ and $q^*$ into the equation for K.

$$K = \frac{A\lambda}{\infty}\left[\frac{IC}{2B\lambda}\right]^{\frac{1}{2}} + \left[\frac{B^2\lambda^2 IC}{2B\lambda}\right]^{\frac{1}{2}} + \left[\frac{(IC)^2 2B\lambda}{2^2 IC}\right]^{\frac{1}{2}} ,$$

or

$$K = (2B\lambda IC)^{\frac{1}{2}} .$$

This result is of the same form as the Wilson Q, but has B in place of A. This is not unexpected, since A is spent only once while B is spent as a result of each delivery. $Q = nq$ is infinite and it seems quite reasonable that the Wilson formula apply to each increment rather than to the infinite order quantity, Q. However, such an infinite order quantity is obviously not applicable to any real system. Indeed, this model has ignored many realistic constraints. Conditions vary over time and organizations are reluctant to commit themselves over long periods. It is therefore important to consider the results when n is constant (not a decision variable) or constrained. Other constraints will be considered in a later model.

When n is constrained to some maximum value N, the optimal solution from equation (2.2) is for $n^2q$ to be a maximum which occurs when n is equal to its maximum allowed value N. If n is constant (n=N) or restricted to finite values, $(n \leq N)$ then

$$q^* = \left[\frac{2\lambda(A+NB)}{NIC}\right]^{\frac{1}{2}} ,$$

$$n^* = N ,$$

and

$$Q^* = q^*N = \left[\frac{2N\lambda(A+NB)}{IC}\right]^{\frac{1}{2}} .$$

The optimal value for K is found by substituting n* and q* into the equation for K.

$$K^* = \left[\frac{A^2\lambda^2ICN}{N^2 2\lambda(A+NB)}\right]^{\frac{1}{2}} + \left[\frac{B^2\lambda^2NIC}{2\lambda(A+NB)}\right]^{\frac{1}{2}} + \left[\frac{I^2C^2 2\lambda(A+BN)}{4NIC}\right]^{\frac{1}{2}} ,$$

or

$$K^* = \left[\frac{2IC\lambda(A+NB)}{N}\right]^{\frac{1}{2}}$$

This value of K* is always less than the conventional $K_W$ since (A+NB)/N is less than A' = A + B, the order cost for the Wilson Q model.

If B = 0 then

$$q^* = \frac{Q_W}{N^{\frac{1}{2}}} ,$$

$$Q^* = N^{\frac{1}{2}}Q_W ,$$

and

$$K^* = \frac{K_W}{N^{\frac{1}{2}}} .$$

These results appear qualitatively reasonable since holding cost is reduced relative to the Wilson Q formulation by the incremental delivery schedule, while with B = 0, order cost is unchanged. It can be seen that this will increase total order quantity but require q be less than $Q_W$ and result in an optimal cost, K*, which is less than $K_W$, the Wilson cost.

## 2.3 Comparison with Finite Production Rate Model

It is also informative to note the relationship of this model to the finite production rate model. Consider the optimal n given fixed values for q and $\tau$. Recall $R = q/\tau$ is the discrete analogue of $\Psi$ and that $R \rightarrow \Psi$ as $\tau$ and $q \rightarrow 0$ and $n \rightarrow \infty$. Also recall

$$K = \frac{A\lambda}{nq} + \frac{B\lambda}{q} + \frac{IC}{2}(qn - n\lambda\tau + \lambda\tau) \ .$$

If we let B equal zero then

$$\frac{\partial K}{\partial n} = \frac{-A\lambda}{n^2 q} + \frac{IC}{2}(q - \lambda\tau) = 0 \ .$$

Solving for n we find that

$$n^* = \left[\frac{2A\lambda}{ICq(q-\lambda\tau)}\right]^{\frac{1}{2}} ,$$

since

$$\frac{\partial^2 K}{\partial n^2} = \frac{+2A\lambda}{n^3 q} > 0$$

implies that a minimum exists at the solution point. Now $Q = nq$ so that

$$Q^* = qn^* = \left[\frac{2A\lambda}{IC}\,\frac{q}{(q-\lambda\tau)}\right]^{\frac{1}{2}} = \left[\frac{2A\lambda}{IC}\,\frac{q/\tau}{(q/\tau-\lambda)}\right]^{\frac{1}{2}} \ .$$

Note that this can be written as

$$Q^* = Q_W \left(\frac{R}{R-\lambda}\right)^{\frac{1}{2}} \ .$$

This result is consistent with the finite production model with R replacing $\Psi$. It is exactly the same result in the limit as $\tau$ and q go to zero and n goes to infinity.

# CHAPTER 3

# CONSTRAINED PHASED DELIVERY MODELS WITH EQUAL DELIVERY LOT SIZE AT EQUAL INTERVALS

## 3.1 General Discussion

It is obvious from the preceding model that an unconstrained incremental delivery model is applicable only when the manager has very extensive control over the system and knowledge of the present and future values of the parameters involved. Although the inventory manager will very seldom have such complete control of the delivery process, he will usually have some degree of control or a set of choices he can exercise.

The following phased delivery models assume varying degrees of control, and apply realistic constraints to the system. The models have increased management control in order of presentation. The first section actually contains two models. We first assume $\tau$ and q are fixed constants set by the manufacturer (only n is variable) and then assume that the production (delivery) rate $q/\tau$ is fixed as a ratio but that subject to $q/\tau$ equals R we can vary q, $\tau$, and n. This is a restricted 2 variable model. The next model in the series fixes only the interval $\tau$ between deliveries and leaves q and n as variables. The final model is a three variable model with $\tau$, q and n restricted only by inequality constraints.

These models are deterministic models designed for application wherever expected values can be used in lieu of random variables for demand, leadtime, and order quantity. Shortages will not be allowed in these models. Probabilistic considerations and a discussion of shortages will be considered in sections 5.1 and 5.2.

The models are similar to the model of section 2.2, but the effect of the varying degree of management control and the constraints is to limit the search for optimal operating policies to an area of feasible solutions. This helps to insure applicable, implementable results.

Some discussion of desirable constraints is necessary. The most obvious constraints are that n and q must be positive integers. For the reasons discussed earlier, we will not require n and q to be integers but the constraints $n \geq 1$ and $q \geq 1$ prove useful. A constraint on $\tau$ of the form $\tau \geq a$ puts an upper bound on delivery frequency which may be necessary to prevent an unreasonable workload on the receiving department. Direct constraints on q of the form $q \leq M$ may be necessary to prevent overloading of warehouses or to keep the shelf time down to prevent spoilage. Constraints of the form $q \geq m_1$ may be necessary to insure carload lot size shipping economies or to keep the value above some minimum order size. For simplification, recall that $q \geq \lambda\tau$, so $\tau \geq a$ implies $\lambda\tau \geq \lambda a = m_2$ which is equivalent to $q \geq m_2$. If we have several constraints of the form $q \geq m_i$, we can combine them into one constraint $q \geq m$ where m is the maximum of the $m_i$. In this way constraints on $\tau$ and q can be consolidated into a pair of constraints $q \geq m$ and $q \leq M$.

In a real system uncertainty always exists and there is always an explicit or implied planning horizon which may not be exceeded. In this context, a constraint on maximum time for use of ordered assets is imposed. This constraint may be applied in the form $Q/\lambda \leq T$ or $nq \leq \lambda T$. Obviously the time constraint is exactly equivalent to a constraint on maximum order quantity $nq = Q$ applied directly. (i.e. $Q \leq \frac{\lambda}{T} = W_1$) A constraint on maximum order value would be exactly equivalent in form. If the maximum order quantity must be less than X dollars, $QC \leq X$ is exactly equivalent to $Q \leq \frac{X}{C} = W_2$. If several such constraints were to be applied, they could all be put into the form $Q \leq W_i$ and the single constraint, $Q \leq W = \min_i (W_i)$ insures all are met. This constraint, $Q = nq \leq W$, is nonlinear in terms of the decision variables, n and q. Similarly, a nonlinear constraint of the form, $w \leq Q = nq$, could be imposed to keep down the workload on the purchasing department.

An upper bound on n, which was applied earlier, is rather artificial, since the constraints on $\tau$ and Q probably represent the true motivation for constraining n. Note that the constraints used imply $1 \leq n \leq \frac{W}{m}$.

A constraint on maximum on hand quantity would be a constraint of the form $n(q-\lambda\tau) \leq z$. This constraint is not imposed here to prevent overcomplicating the model, but it is obvious that the constraints imposed define an upper bound on the maximum stock on hand. Since $nq = Q \leq W$, $n \geq 1$ and $\lambda > 0$ it is obvious that for $n = 1$ the maximum on hand is $Q \leq W$, immediately after receipt of the quantity Q. If $n > 1$ then $\tau > 0$ and the maximum on hand quantity will be $n(q-\lambda\tau) = Q - n\lambda\tau \leq W - \lambda\tau \leq W$. The maximum on hand quantity occurs immediately after the nth receipt. When n is greater than one, the maximum on hand will probable be very much less than W.

### 3.2 Delivery Size and Interval Fixed

Suppose the source of material offers shipments of 200 on the first of each month. Now q and $\tau$ are given constant parameters and our only decision variable is n. Recall that for feasibility, q must be greater or equal to $\lambda\tau$. See Figure 4 for a graphic representation of a cycle.

From the prior model we have total variable cost

$$K = \frac{A\lambda}{nq} + \frac{B\lambda}{q} + \frac{IC}{2}\,(nq-n\lambda\tau+\lambda\tau).$$

Since n is the only decision variable, the variable cost K can be rewritten as:

$$K = \frac{A\lambda}{nq} + \frac{IC}{2}\,(q-\lambda\tau)n$$

To find the optimal n, we set the derivative of K with respect to n equal to zero and solve for n, checking to

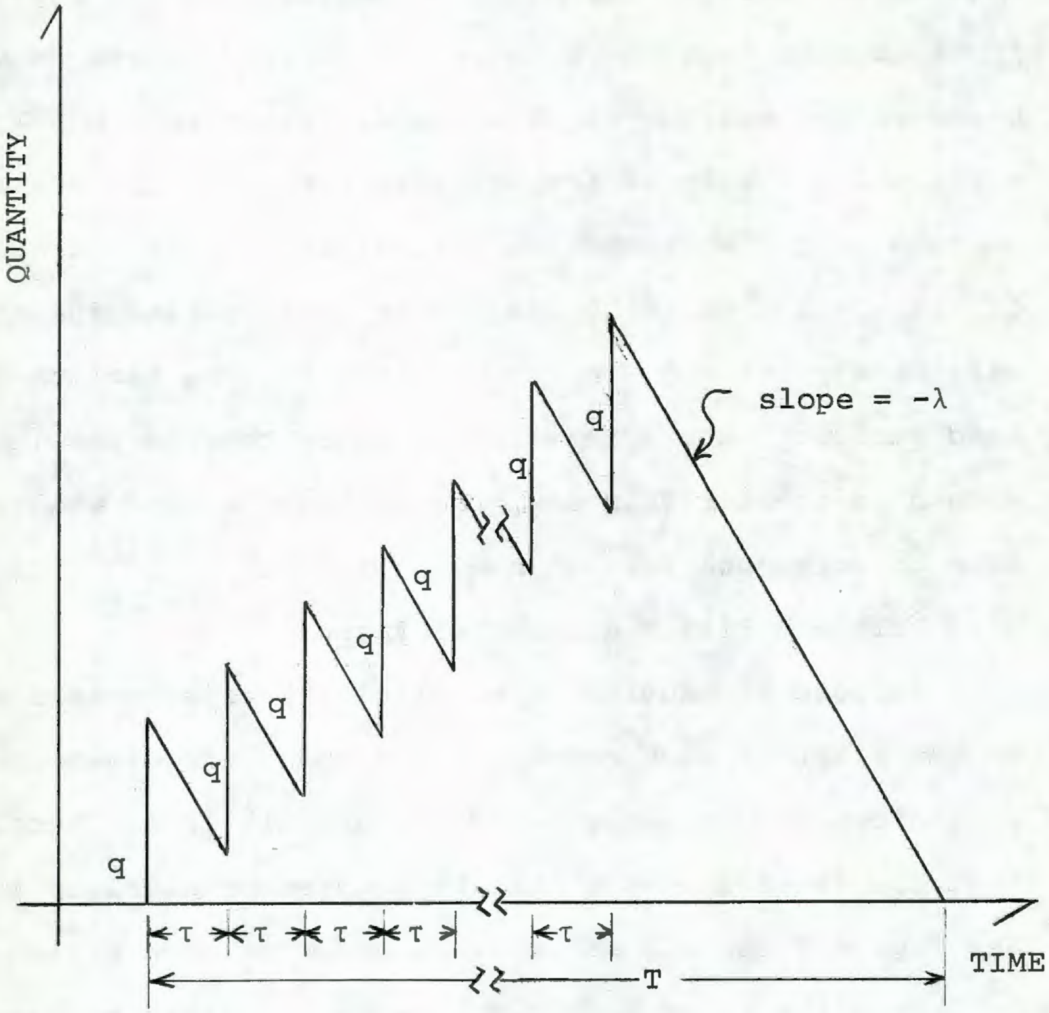


FIGURE 4

DELIVERY SIZE AND INTERVAL FIXED

insure the second derivative of K with respect to n is positive to insure a minimum. Now,

$$\frac{\partial K}{\partial n} = \frac{-A\lambda}{n^2 q} + \frac{IC}{2}(q-\lambda\tau) = 0$$

implies

$$n' = \left[\frac{2A\lambda}{ICq(q-\lambda\tau)}\right]^{\frac{1}{2}} .$$

Also

$$\frac{\partial^2 K}{\partial n^2} = \frac{+2A\lambda}{n^3 q} \geq 0$$

for all $n \leq \infty$.

There will be no problem with imaginary roots since $q \geq \lambda\tau$ and all parameters are strictly positive. If $n' \leq 1$, then $n^* = 1$ and $\tau = 0$, otherwise $n^* = n'$.

If $n^* = n' \geq 1$, then substituting $n^*$ in the equation for K we get

$$K^* = \left[\frac{n^2\lambda^2}{q^2}\,\frac{ICq\,(q-\lambda\tau)}{2A\lambda}\right]^{\frac{1}{2}} + \frac{B\lambda}{q} + \left[\frac{(IC)^2(q-\lambda\tau)^2 2A\lambda}{4\quad ICq(q-\lambda\tau)}\right]^{\frac{1}{2}} + \frac{IC\lambda\tau}{2},$$

or

$$K^* = \left[2A\lambda IC\,\frac{(q-\lambda\tau)}{q}\right]^{\frac{1}{2}} + \frac{B\lambda}{2} + \frac{IC\lambda\tau}{2},$$

which may be greater or less than $K_w$, depending upon the values of the input parameters. The manager has insufficient control over the situation to guarantee a cost less than $K_w$.

If $n' \leq 1$ then $n^* = 1$, $\tau = 0$ and

$$K = \frac{(A+B)\lambda}{q} + \frac{ICq}{2} \geq K_w$$

and is equal to $K_w$ if and only if q is equal to $Q_w$.

Before passing on to the next point, it is worth noting that if $q = \lambda\tau + \varepsilon$ then $\lim_{\varepsilon \to 0} n^* = \infty$. Also

$$K^* = \sqrt{\frac{2A\lambda IC\varepsilon}{q}} + \frac{B\lambda}{q} + \frac{IC(q+\varepsilon)}{2},$$

and

$$\lim_{\varepsilon \to 0} K^* = \frac{B\lambda}{q} + \frac{ICq}{2},$$

which is of the form of $K_w$ and is minimal when

$$q = \sqrt{\frac{2B\lambda}{IC}}.$$

Now let us suppose that the manufacturing source offers shipments at any intervals but will ship all items produced since the last shipment. That is $q = \Psi\tau$ where $\Psi$ is the production rate. $q$ is a decision variable but $\tau$ is restricted by the relationship $q = \Psi\tau$, so we can substitute $q/\Psi$ for $\tau$. For feasibility $\Psi \geq \lambda$. Then

$$K = \frac{A\lambda}{nq} + \frac{B\lambda}{q} + \frac{IC}{2}\left(nq - nq\frac{\lambda}{\Psi} + \frac{q\lambda}{\Psi}\right).$$

Now we take the derivatives of K with respect to n and q and set them equal to zero to solve for the optimal values.

$$\frac{\partial K}{\partial q} = \frac{-A\lambda}{nq^2} \frac{-nB\lambda}{nq^2} + \frac{IC}{2}\left(n - \frac{n\lambda}{\Psi} + \frac{\lambda}{4}\right) = 0$$

implies

$$q^2 = \frac{2\lambda(A+nB)}{nIC(n - \frac{n\lambda}{\Psi} + \frac{\lambda}{\Psi})}.$$

$$\frac{\partial K}{\partial n} = -\frac{A\lambda}{n^2 q} + \frac{ICq}{2}\left(1 - \frac{\lambda}{\Psi}\right) = 0$$

implies

$$n^2 = \frac{2A\lambda}{ICq^2}\left(\frac{\Psi}{\Psi-\lambda}\right) ,$$

or

$$q^2 = \frac{2A\lambda}{ICn^2}\left(\frac{\Psi}{\Psi-\lambda}\right) .$$

The Hessian is always positive definite, so that we are assured of having a minimum. Setting the two equations for $q^2$ in terms of n equal to each other, we get n* in terms of the invariant parameters, which can be substituted into either equation to find q*. We now have

$$n^* = \left[\frac{A}{B}\left(\frac{\lambda}{\Psi-\lambda}\right)\right]^{\frac{1}{2}} ,$$

and

$$q^* = \left[\frac{2\Psi B}{IC}\right]^{\frac{1}{2}} ,$$

which imply

$$\tau^* = \frac{q^*}{\Psi} = \left[\frac{IB}{\Psi IC}\right]^{\frac{1}{2}} .$$

It is probably not surprising to see that

$$Q^* = q^*n^* = \left[\frac{2A\lambda}{IC}\left(\frac{\Psi}{\Psi-\lambda}\right)\right]^{\frac{1}{2}} ,$$

which is the same result as for the finite production rate model of the conventional literature (see Section 1.3). This means that we can decide on our total order quantity in the usual fashion and then ex post facto decide on our delivery schedule using the preceding model.

However, it is probably most surprising to find q*, the optimal delivery lot size, is a function of $\Psi$, the production rate, and not $\lambda$, the demand rate. It has the usual form for an optimal order size formula otherwise. The $\lambda$ dependency is built into the optimal value of n*. Perhaps the best intuitive explanation for this is that the choice of q* and $\tau$* is forced on us by the delivery condition and our demand rate must exert its influence through the value of n.

In a draft Army Material Command report, Deemer and Hoekstra [3] consider a probabilistic version of this model. They simplify the model by assuming that $\tau$ is always fixed at one month and that delivery quantity is not only constant, but exactly equal to $\lambda\tau$, the expected monthly demand. The probabilistic nature of demand implies an increasing risk of stock out, since the expected safety stock remains constant throughout the contract period. This results in a high protection level during the early part of the contract period, and lower protection late in the cycle. It would appear better to choose a delivery lot size somewhat in excess of $\lambda\tau$. This will result in the expected value of the safety level increasing as the demand variability increases. For a given expenditure for holding safety stock, we would be able to maintain a fixed protection level, or make an appropriate trade off between rising risk and rising stock levels.

For example, consider a one year contract period with monthly deliveries. Assume demand is normally distributed with a time dependent standard deviation $\sigma = t$, where t is in months. We decide to hold 156 unit months of safety stock. If we establish a constant safety level of 13 units with q equal to the expected monthly demand, then the average protection level is

$$1/12\ (1.0 + 1.0 + 1.0 + .9994 + .9953 + .9849 + .9684 + .9479 + .9257 + .9032 + .8814 + .8606) = .9639 \quad .$$

If we establish an increasing safety level by starting with a safety level of two and let q be equal to the expected monthly demand plus two units, the safety level during the ith month will be 2i. Since the standard deviation in the ith month is i, the average protection level will be constant at .9773.

In this example, we can increase the expected performance with the same average safety level by using an order quantity greater than the monthly demand and allowing the safety level to build up during the contract period.

### 3.3 Delivery interval fixed

In this model it is assumed that $\tau$ is fixed, while n and q are decision variables. This type of problem would arise whenever shipping schedules were externally set. Actually there may be a choice of several values for $\tau$. For example, a heavy item may be transported by ship with a twice a month schedule. Then our choices for $\tau$ are ½ month, 1 month, 1 ½ months, etc. An item which is batch produced may have a 4 month batch processing time. If only one batch can be produced at a time, we could, under certain conditions, model this with $\tau$ equals 4 months, 8 months, etc. In some applications, management policy may specify a $\tau$ which is based on factors exogenous to the model. Top management may specify monthly deliveries of raw materials based on a combination of shipping schedules, storage capacity, and the extra workload due to more frequent deliveries. If more than one value of $\tau$ is possible, the solution procedure must be repeated for each value of $\tau$ and the proper value of $\tau$ chosen by comparison of the costs of the resulting optimal operating policies. It will be seen later that only part of the computation need be recalculated.

Recall q must be greater or equal to $\lambda\tau$ for feasibility and

$$K = \frac{A\lambda}{nq} + \frac{nB\lambda}{nq} + \frac{IC}{2}(nq-n\lambda\tau+\lambda\tau) \quad . \tag{3.1}$$

We follow the usual procedure.

$$\frac{\partial K}{\partial q} = \frac{-A\lambda}{nq^2} - \frac{nB\lambda}{nq^2} + \frac{ICn}{2} = 0$$

implies

$$q = \frac{1}{n}\left[\frac{2\lambda(A+nB)}{IC}\right]^{\frac{1}{2}} \quad . \tag{3.2}$$

This describes a "valley" in the n,q plane's first quadrant for which the value of q is optimal given any value for n. Note that when n is equal to 1 the optimal value of q is $Q_W$. To confirm that the valley is a valley of minimums, not a ridge of maximums, we investigate the second derivative.

$$\frac{\partial^2 K}{\partial q^2} = \frac{A\lambda}{nq^3} + \frac{nB\lambda}{nq^3} > 0 \quad .$$

Now substituting equation (3.2) into equation (3.1), we get the cost as a function of n only, with optimal (minimum cost) q chosen for each n by equation (3.2). The equation for the cost in the "valley" is

$$K = \frac{\lambda(A+nB)(IC)^{\frac{1}{2}}}{[2\lambda(A+nB)]^{\frac{1}{2}}} + \frac{IC}{2}\left[\frac{2\lambda(A+nB)}{IC}\right]^{\frac{1}{2}} - \frac{n\lambda\tau IC}{2} + \frac{IC\lambda\tau}{2}$$

$$= (2\lambda IC)^{\frac{1}{2}}(A+nB)^{\frac{1}{2}} - (n-1)\frac{\lambda\tau IC}{2} \quad .$$

Now to find the optimal value of n we consider the derivative

$$\frac{\partial K}{\partial n} = \frac{(2\lambda IC)^{\frac{1}{2}}B}{2(A+nB)^{\frac{1}{2}}} - \frac{\lambda\tau IC}{2} = 0 \ .$$

Solving for n we get

$$n = \frac{2B}{\lambda IC\tau^2} - \frac{A}{B} \quad ,$$

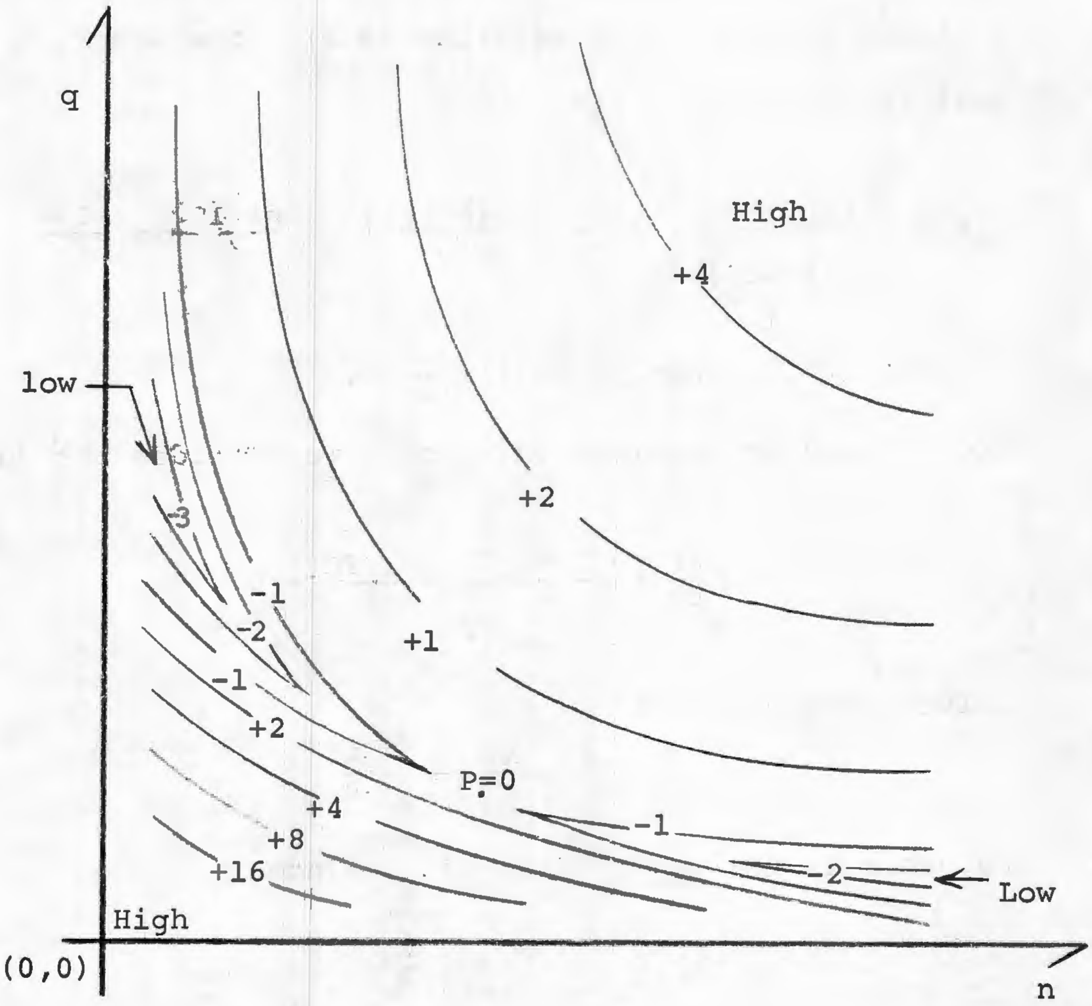
but this is the worst value of n, since

$$\frac{\partial^2 K}{\partial n^2} = \frac{-(2\lambda IC)^{\frac{1}{2}}B^2}{4(A+nB)^{\frac{3}{2}}} < 0 \ ,$$

implies we have a maximum. That is the valley has a high point and decreases toward the right and left from the point where

$$n = \frac{2B}{\lambda IC\tau^2} - \frac{A}{B} \ .$$

This value may be positive or negative and the portion of the "valley" we are interested in may slope downward to the right or left or to both right and left. Figure 5 shows a contour map of a typical K over the first quadrant of the n, q plane. The unconstrained minimum is at n = ∞ where the "valley" is



Note: The "valley" is between two arrows marked low. Point P is the highest point in the "valley".

FIGURE 5

CONTOUR MAP OF K OVER n, q PLANE

asymtotic to the $q = 0$ line. This is consistent with the results of the initial phased delivery model of section 2.2.

Now we introduce the following constraints

$$n \geq 1 ,$$

$$q \geq m ,$$

and

$$w \leq nq = Q \leq W .$$

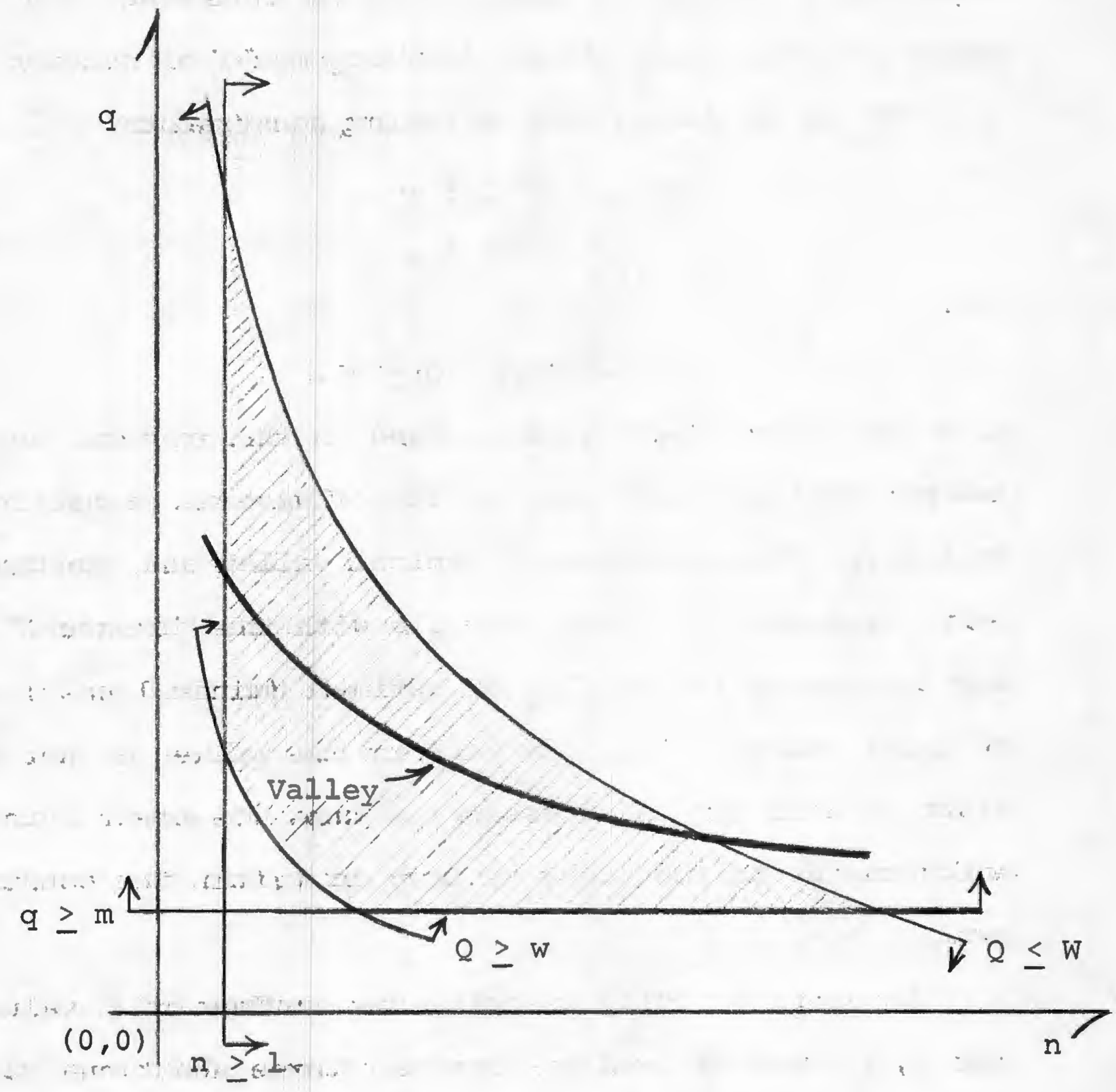
W, w and m are chosen as discussed in the previous section and may represent the most strict of several verbalized constraints. Figure 6 shows a typical valley and constraint set. Feasible solutions must lie with the "crescent" area and the valley is the line of optimal (minimal cost) values of q for values of n. The peak in the valley is not shown, since it will not be necessary to know its exact location which may be to the right or left or within the constraint set.

In order to fully describe the surface of K values over the n, q plane we need to consider three additional things. First, how does K behave on a constraint boundary with $Q = nq$ fixed? To find out we substitute Q for nq in equation (3.1) to get

$$K = \frac{A\lambda}{Q} + \frac{nB\lambda}{Q} + \frac{ICQ}{2} - \frac{nIC\lambda\tau}{2} + \frac{IC\lambda\tau}{2} .$$

The only variable terms are $\frac{nB\lambda}{Q}$ and $\frac{-nIC\lambda\tau}{2}$.

Therefore, K is monotonically increasing or decreasing with respect to n dependent upon whether $\frac{B}{Q} - \frac{IC\tau}{2}$ is positive or negative. The second consideration is the behavior of K over a constraint boundary with $q = m$. Substitute the constant m



Note: Shaded area is feasible solution set.

FIGURE 6

TYPICAL "VALLEY" AND FEASIBLE SOLUTION SET for q in equation (3.1) to get

$$K = \frac{A\lambda}{nm} + \frac{B\lambda}{m} + \frac{ICnm}{2} - \frac{ICn\lambda\tau}{2} + \frac{IC\lambda\tau}{2} .$$

For feasibility $m \geq \lambda\tau$. If $m = \lambda\tau$ then K decreases monotonically as n increases. If $m \geq \lambda\tau$ the terms of K which depend on n are of the form $C_1/n + C_2 n$ where $C_1$ and $C_2$ are positive constants.

$$C_1 = \frac{A\lambda}{m}$$

and

$$C_2 = \frac{IC}{2}(m-\lambda\tau).$$

This implies that K has a minimal value for some n such that $0 < n^* < \infty$. Specifically

$$n^* = \left[\frac{2A\lambda}{ICm(m-\lambda\tau)}\right]^{\frac{1}{2}} .$$

Unfortunately, this point is not necessarily in the "valley". Finally, a quick look at equation (3.2) in the form

$$Q = nq = \left[\frac{2\lambda(A+nB)}{IC}\right]^{\frac{1}{2}}$$

shows that the "valley" has higher values of Q for higher values of n, since n is always positive. This insures that the "valley" rises, relative to lines of constant Q value, such as the non-linear boundaries formed by the Q constraints.

We now have sufficient facts to determine the optimal solution. Let us marshall the facts in order to see clearly where the optimal values lie.

First, for any given value of n, the valley contains the unconstrained optimum value of q. The value of K as a function of Q only is monotonically increasing as we move away from the valley. Therefore, for any value of n for which the valley is outside the feasible solution set, the optimal q is on the boundary "nearest" the valley.

Second, the valley slopes downward toward at least one end, and therefore the section of valley within the feasible solution set has its minimum at a boundary.

Third, the first two arguments insure that the optimal values of q and n lie on a boundary constraint.

Fourth, since we know that along constraints of constant Q value K monotonically increases to either the right or left, the optimum cannot be on such a constraint.

Now we can conclude that the optimum feasible solution is on the constant $n = 1$ with $w \le q \le W$ and $q \ge m$, or on the constraint $q = m$ with $w/m \le n \le W/m$, and $n \ge 1$.

We will need to compute and compare the cost of two solutions to find the optimal policy. First we consider the candidate optimum with $n' = 1$, and

$$q = \left[\frac{2\lambda(A+B)}{IC}\right]^{\frac{1}{2}}.$$

If $q < \max(w,m)$ then $q' = \max(w,m)$. If $q > W$ then $q' = W$ otherwise $q' = q$. Now compute $K'$ for $n'$ and $q'$. Then compute the other candidate optimum with $q'' = m$, and

$$n = \left[\frac{2A\lambda}{ICm(m+\lambda r)}\right]^{\frac{1}{2}}.$$

If $n < \max(1, w/m)$ then $n'' = \max(1, w/m)$. If $n > W/m$ then $n'' = W/m$. Otherwise $n'' = n$. Now compute $K''$ for $n''$ and $q''$.

The minimum of $K'$ and $K''$ is $K^*$ and $n^*$ and $q^*$ are the corresponding n and q values.

It is interesting to note that, in general, the optimal policy is either the conventional single increment policy, or else the opposite extreme; order the minimum allowed increment size and the maximum allowed total quantity. However, if the Q constraints are severe or m is comparable in magnitude to $\lambda$ and $\tau$ is very small, this comment may not apply.

### 3.4 Full Control of Delivery Size and Interval

Auume we have full control over n, $\tau$, and q. Based on prior models, we anticipate the need for a constraint on maximum order size. That is, we will require nq to be less than or equal to some maximum order size M. Since this model is deterministic, we will have each cycle identical. Now order cost per unit time is

$$\frac{A\lambda}{nq} + \frac{B\lambda}{q},$$

and holding cost per unit time is

$$\frac{IC}{2}(nq - n\lambda\tau + \lambda\tau),$$

so that

$$K = \frac{A\lambda}{nq} + \frac{B\lambda}{q} + \frac{IC}{2}(nq - n\lambda\tau + \lambda\tau).$$

Let us first consider the optimal value of $\tau$.

$$\frac{\partial K}{\partial \tau} = \frac{IC}{2}(1-n)\lambda \leq 0$$

for all $n \geq 1$.

This implies K is linear in $\tau$ and inspection shows that K decreases as $\tau$ increases. However, for feasibility $q \geq \lambda\tau$ so that we can determine $\tau^*$ in terms of $q^*$. That is, we have reduced the model to two variables, and $\tau^* = q^*/\lambda$. This allows us to substitute $q/\lambda$ for $\tau$ and K becomes

$$K = \frac{A\lambda}{nq} + \frac{B\lambda}{q} + \frac{IC}{2}q .$$

Now we see that since K decreases as nq increases and $nq \leq M$ $n^*q^*$ will be equal to M, and we can determine $n^*$ in terms of $q^*$. We can now substitute M/q for n and K becomes

$$K = \frac{A\lambda}{M} + \frac{B\lambda}{q} + \frac{IC}{2}q .$$

The first term is a constant, so the variable cost K which we wish to minimize is

$$K = \frac{B\lambda}{q} + \frac{IC}{2}q .$$

and we see that we have reduced the model to a single decision variable, q. This is immediately recognizable as being of the simple lot size form with

$$q^* = \sqrt{\frac{2B\lambda}{IC}} .$$

Interpretation of the optimal policy is now simple and straight foward. The optimal lot size is computed, using the Wilson simple lot size formula with only the B cost term used as "order cost". A is spent at intervals of $M/\lambda$

and its cost contribution is determined by the external policy decision never to order more than M. The optimal interval between deliveries is determined by the relationship $\tau^* = q^*/\lambda$. The optimal total order is M and therefore $n^* = M/q^*$.

# CHAPTER 4

# CONSTRAINED, PHASED DELIVERY MODELS WITH UNEQUAL DELIVERY LOTS AND UNEQUAL INTERVALS

## 4.1 General Discussion

The main application of unequal delivery increment or interval assumptions is to model single period non-steady state systems. Such delivery, obviously, could never be truly optimal in a steady state. However, we may want to accept unequal delivery increments to minimize lead time when producer has an increasing production rate as a result of having a partial quantity on hand, or applying additional resources to the item's production as other contracts are phased out. While the following models presume such an increasing rate, it would be relatively easy to model a decreasing rate if desired. The first model makes no assumption as to the relative size of delivery increments.

These models have limited applicability and the more basic question of variable production rates is taken up in section 7.1. The first and most general of these models is considered in some detail and shortages are allowed. The latter two models are formulated and a solution technique is given but are not fully explored since the assumptions are exceedingly restrictive.

It is important to recall that in steady state systems the optimal procedures result in repetitive identical cycles, and any forced deviation from identical replication will be nonoptimal. That is, given completely free choice, one would never voluntarily choose unequal delivery increments or intervals except as a response to probabilistic influences or changing parameters. This means that basically this set of models should only be applied when outside conditions require their use.

## 4.2 Fractional Delivery at Fixed Unequal Intervals

This series of models is an outgrowth and extension of an exercise in Hadley and Whiten [1].[1] Demand is deterministic. It is initially assumed that no shortages are allowed and that delivery will be made in two increments $q_1$ and $q_2$, which may not be equal in size. More specifically, a fraction $f_1$ of the total quantity ordered, $Q$, will be delivered first, and then after an interval $\tau$ the remaining fraction $f_2 = 1 - f_1$ will be received. Assume temporarily that $\tau < T_w$, the Wilson model optimal cycle time. Now $q_1 = f_1 Q$, $q_2 = f_2 Q$ and $q_1 + q_2 = Q$. Since $n = 2$, $A + nB$ is a constant and $A' = A + 2B$ will be used in this model to represent this total order cost. After the basic model is analyzed, a simple shortage model will be considered for the two part delivery condition, and then the extension of the no shortage model to $n$ delivery segments of size $q_1, q_2, \ldots, q_n$ will be discussed. $Q$ and $b$ are the decision variables. It appears necessary to investigate three cases:

$q_1 = f_1 Q > \lambda\tau \Longleftrightarrow f_1 > \lambda/T$ see Figure 7A

$q_1 = f_1 Q = \lambda\tau \Longleftrightarrow f_1 = \tau/T$ see Figure 7B

$q_1 = f_1 Q < \lambda\tau \Longleftrightarrow f_1 < \tau/T$ see Figure 7C

---

[1] Chapter 2, problem 2-65.

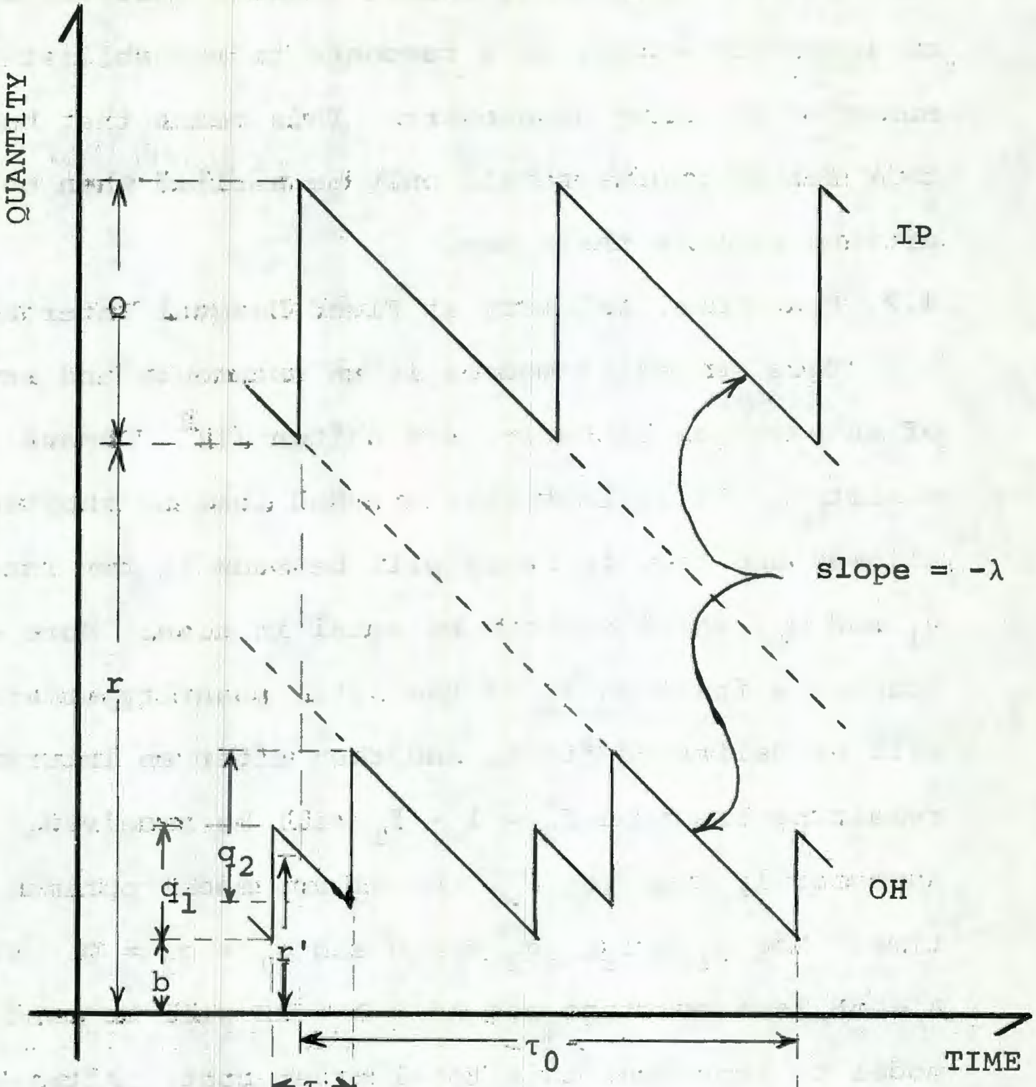


FIGURE 7A

TWO INCREMENT MODEL

$q_1 = f_1 Q > \lambda\tau$

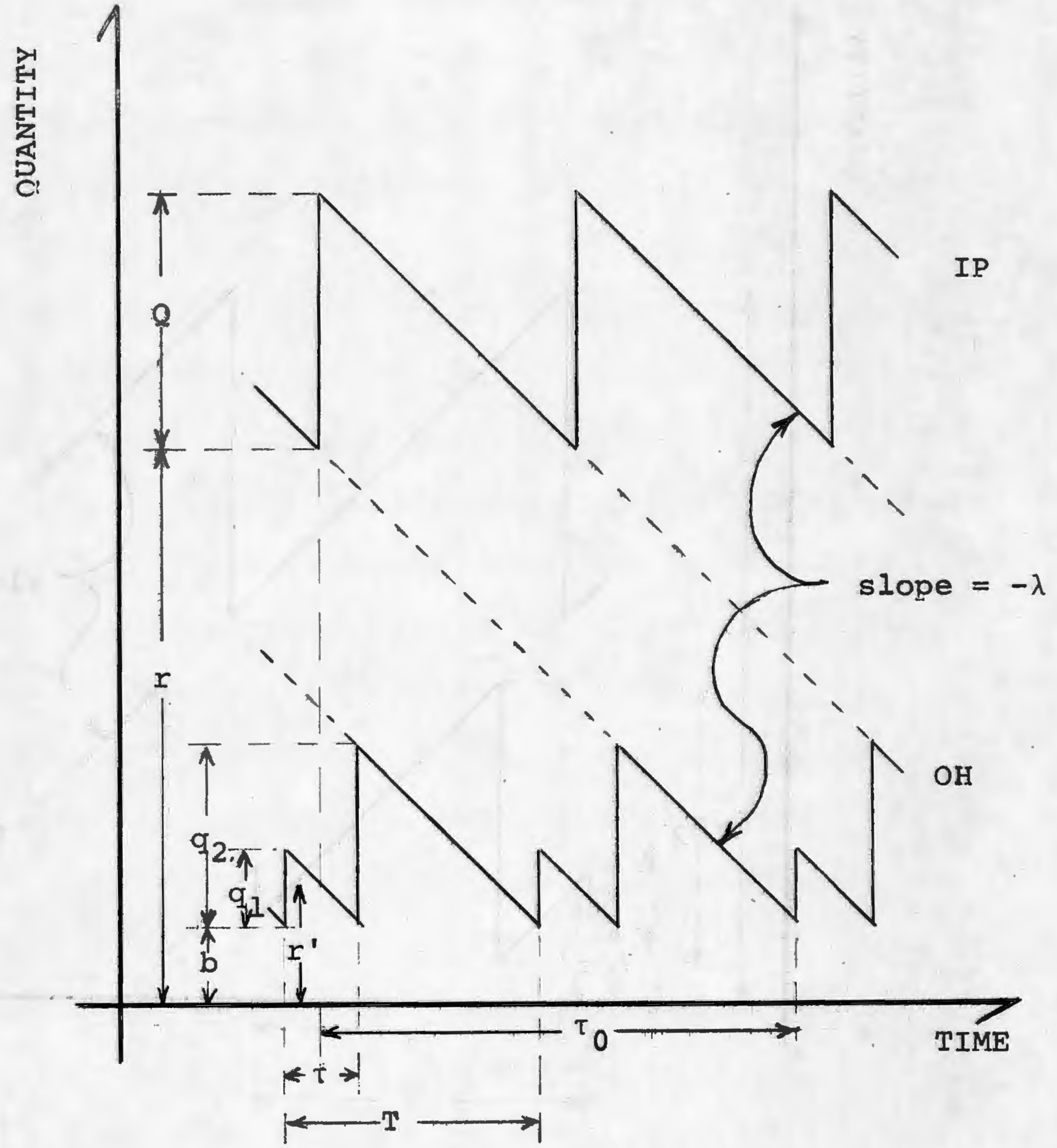


FIGURE 7B

TWO INCREMENT MODEL

$q_1 = f_1Q = \lambda\tau$

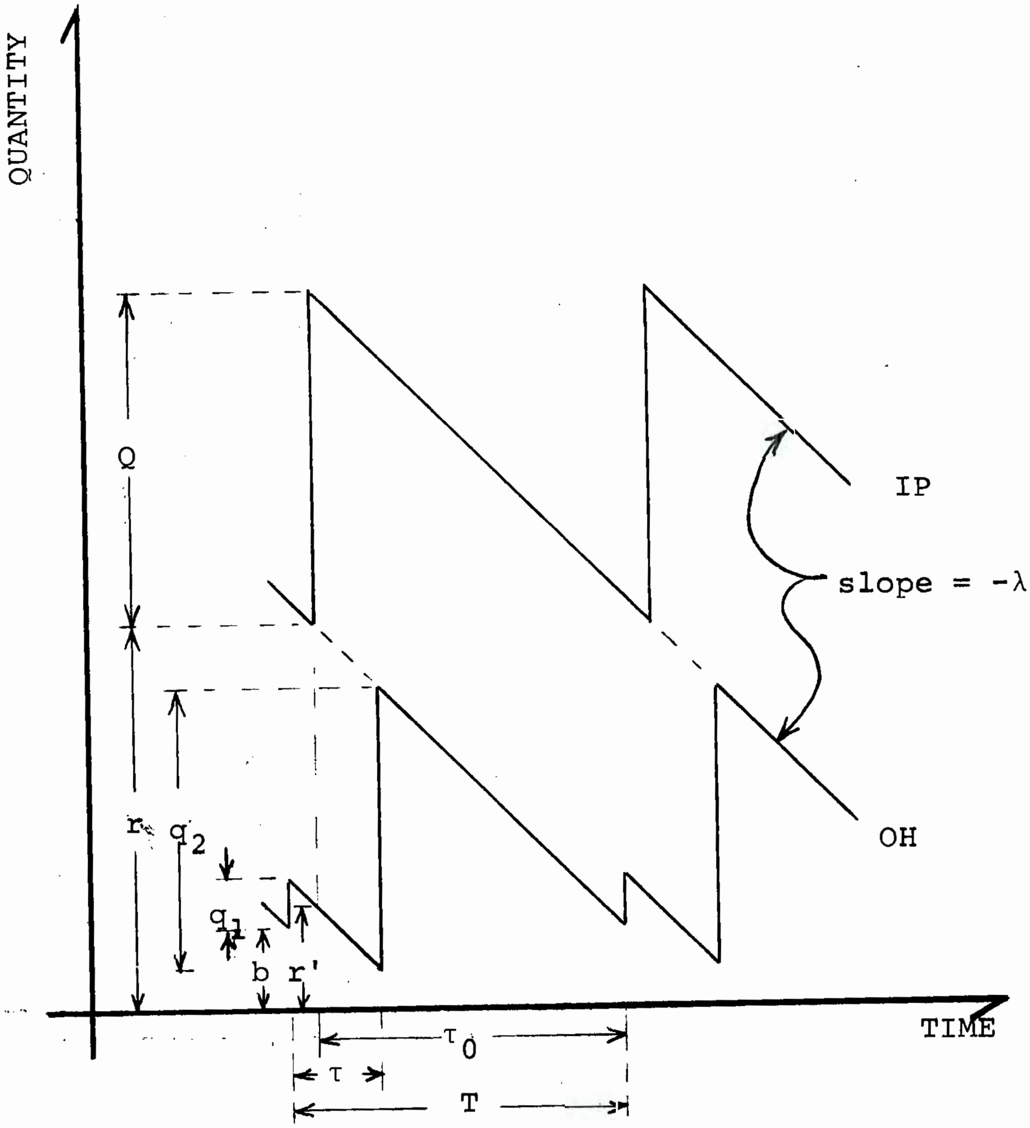


FIGURE 7C

TWO INCREMENT MODEL

$q_1 = f_1 Q < \lambda\tau$

However, it will not be necessary to distinguish between these cases during the first few steps of the analysis, but reference to all three figures will aid in following the analysis. Holding cost per cycle is proportional to the area under the on hand curve. Consider the area under the on hand curve during the time $\tau$. It is a trapezoid with an area of

$$\tau \left[\frac{(b+f_1Q) + (b+f_1Q-\lambda\tau)}{2}\right] .$$

Similarly, the area below the curve between the time $q_2$ is delivered and the end of the cycle is a trapezoid whose area is given by

$$(T-\tau) \left[\frac{(b+\lambda(T-\tau)) + (b)}{2}\right] .$$

Holding cost per cycle is IC times the sum of these two areas, and equals

$$\frac{IC}{2} \left[\tau(2b+2f_1Q-\lambda\tau) + (T-\tau)(2b+\lambda(T-\tau))\right] .$$

or

$$\frac{IC}{2} (2f_1\lambda T\tau+2bT+QT-2\lambda T\tau) .$$

Dividing by T to get holding cost per unit time yields

$$\frac{IC}{2} (2f_1\lambda\tau+2b+Q-2\lambda\tau) ,$$

$$= \frac{ICQ}{2} + IC(f_1\lambda\tau+b-\lambda\tau).$$

Now since order cost per unit time is $A'\lambda/Q$,

$$K = \frac{A'\lambda}{Q} + \frac{ICQ}{2} + IC(f_1\lambda\tau+b-\lambda\tau). \qquad (4.1)$$

Taking the derivatives of K with respect to b and Q, the decision variables, setting the derivatives equal to zero and solving yields,

$$\frac{\partial K}{\partial b} = +IC = 0.$$

This implies K is a linear function of b and inspection of K shows K is minimal when b is minimal. Now,

$$\frac{\partial K}{\partial Q} = \frac{-A'\lambda}{Q^2} + \frac{IC}{2} = 0 ,$$

implies the optimal value of Q is $Q_w$. This is a somewhat surprising result since one would probably expect $Q^* > Q_w$. However, to continue, it is necessary to consider the various cases to determine b*. If $f_1 \geq \tau/T$ then b can equal 0 and $b^* = 0$. Now

$$K^* = \frac{A'\lambda}{Q_w} + \frac{ICQ_w}{2} + IC\lambda\tau(f_1-1),$$

but $f_1 - 1 = -f_2$, so that

$$K^* = K_w - IC\lambda\tau f_2 \leq K_w .$$

However, if $f_1 < \tau/T$, there will be shortages unless $b + f_1Q \geq \lambda\tau$. This implies that $b \geq \lambda\tau - f_1Q$ and the minimal allowable b is the optimal $b^* = \lambda\tau - f_1Q_w$. Now the optimal values b* and Q* can be substituted into the equation for K to get K*;

$$K^* = \frac{A'\lambda}{Q_w} + \frac{ICQ_w}{2} + IC(f_1\lambda\tau+\lambda\tau-f_1Q_w-\lambda\tau)$$

$$= K_w + ICF_1\lambda(\tau-T_w) .$$

But, since we assume $T_w > \tau$ and I, C, $f_1$ and $\lambda$ are all greater than zero, K* can be written so as to make its relationship to $K_w$ obvious;

$$K^* = K_w - ICf_1\lambda(T_w - \tau) \leq K_w .$$

It is now clear that, although $Q^* = Q_w$, the split delivery results in a reduced cost. Under these assumptions, the optimal operating policy is the same as for the simple Wilson model with the addition that $b^* = 0$ or $b^* = \lambda\tau - f_1 Q$ depending on the case occuring.

Actually, only one case is required, as can be readily seen by reference to Figure 8. Since $Q^* = Q_w$ and $T^* = T_w$, the values $q_1$, $q_2$, $\tau$ and $T_w - \tau$ are fixed and the cycle repeats identically. We can now see that we can define a cycle for computation purposes to be $P_1$ to $P_3$ or equally well from $P_2$ to $P_4$. That is, we can choose $f' = f_2$ if and only if $f_1 \geq \tau/T_w$ and $f' = f_1$ otherwise. We then define $\tau' = \tau$ or $\tau' = T_w - \tau$ respectively. Now

$$Q^* = Q_w ,$$

$$b^* = \begin{cases} 0 & \text{if } f_1 \geq \tau/T_w \\ \lambda\tau - f_1 Q_w & \text{if } f_1 < \tau/T_w \end{cases} ,$$

and

$$K^* = K_w - ICf'\lambda\tau' \leq K_w .$$

It is considerably less obvious how to apply the above formulas when $\tau > T_w$. Figures 9A and 9B show cycles graphically for $T_w < \tau \leq 2T_w$ and $2T_w < \tau \leq 3T_w$ . More than one order will be outstanding at times. Define $\tau'' = \tau - nT_w$ such that $T_w \geq \tau'' > 0$. It is clear that $Q^* = Q_w$ is still optimal. It is also readily seen that $\tau''$ can be used in place of $\tau$ to determine $f'$ and $\tau'$ and

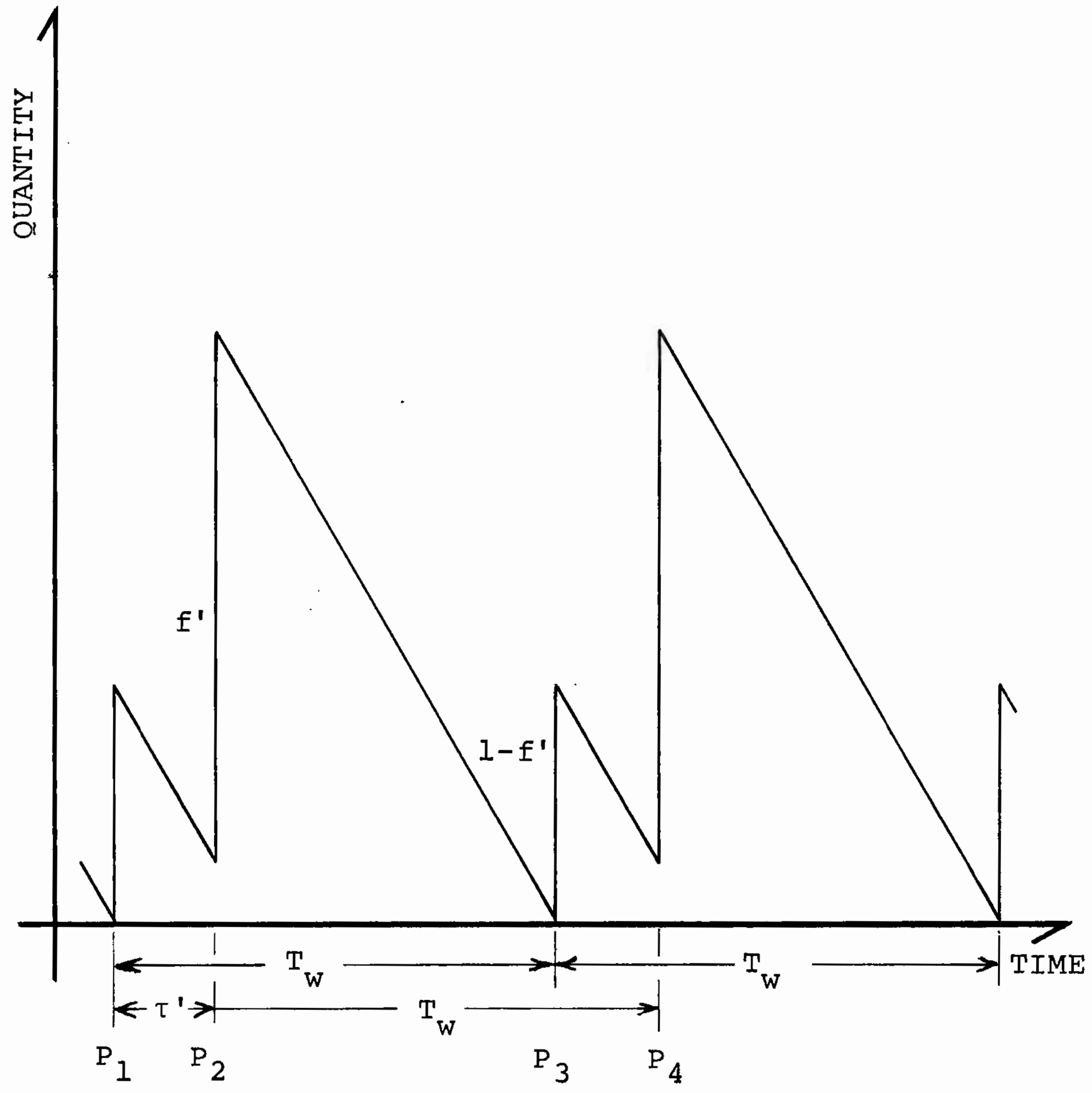


FIGURE 8

SYMMETRY OF TWO INCREMENT MODEL

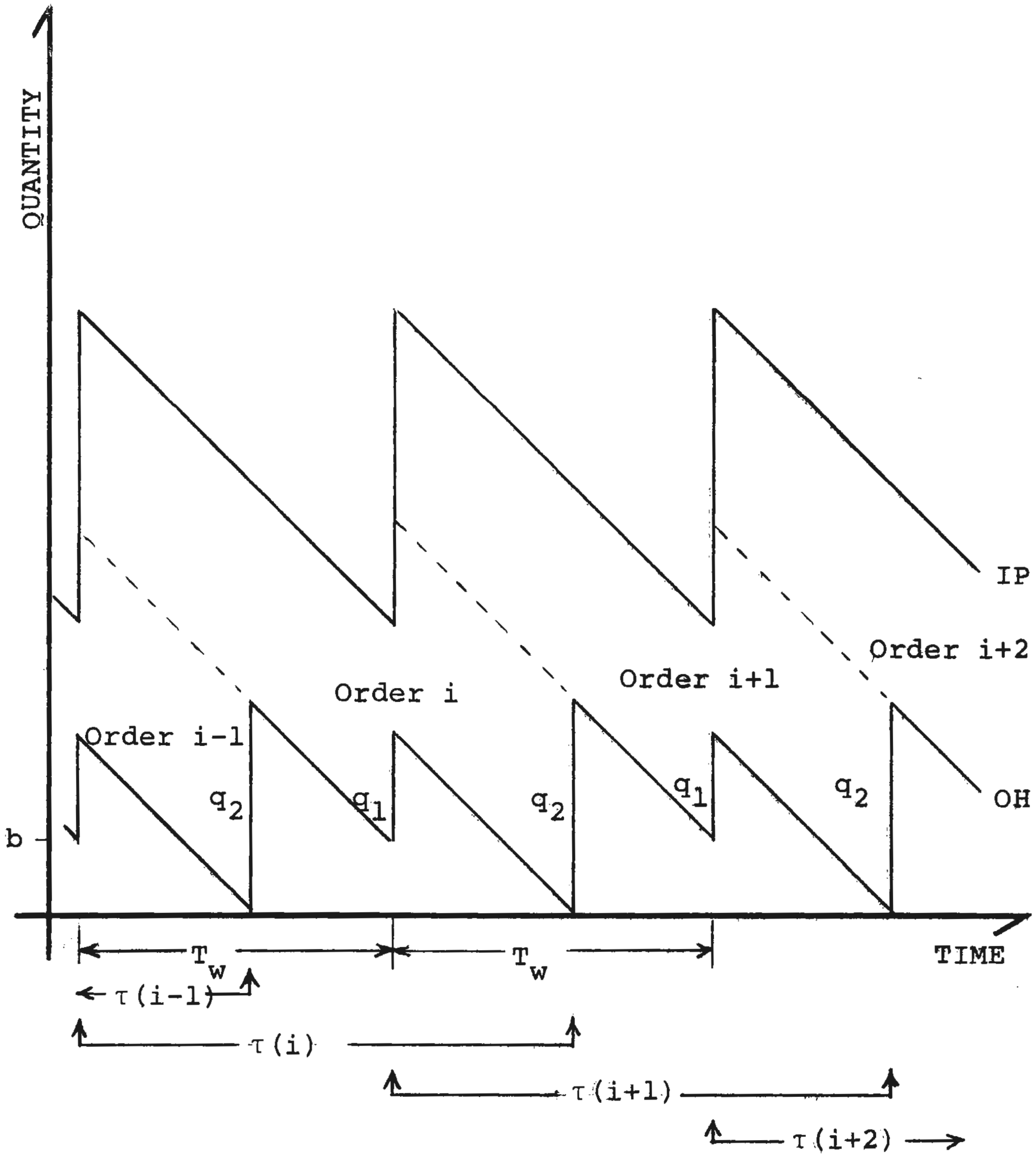


FIGURE 9A

TWO INCREMENT MODEL

$T_w < \tau < 2T_w$

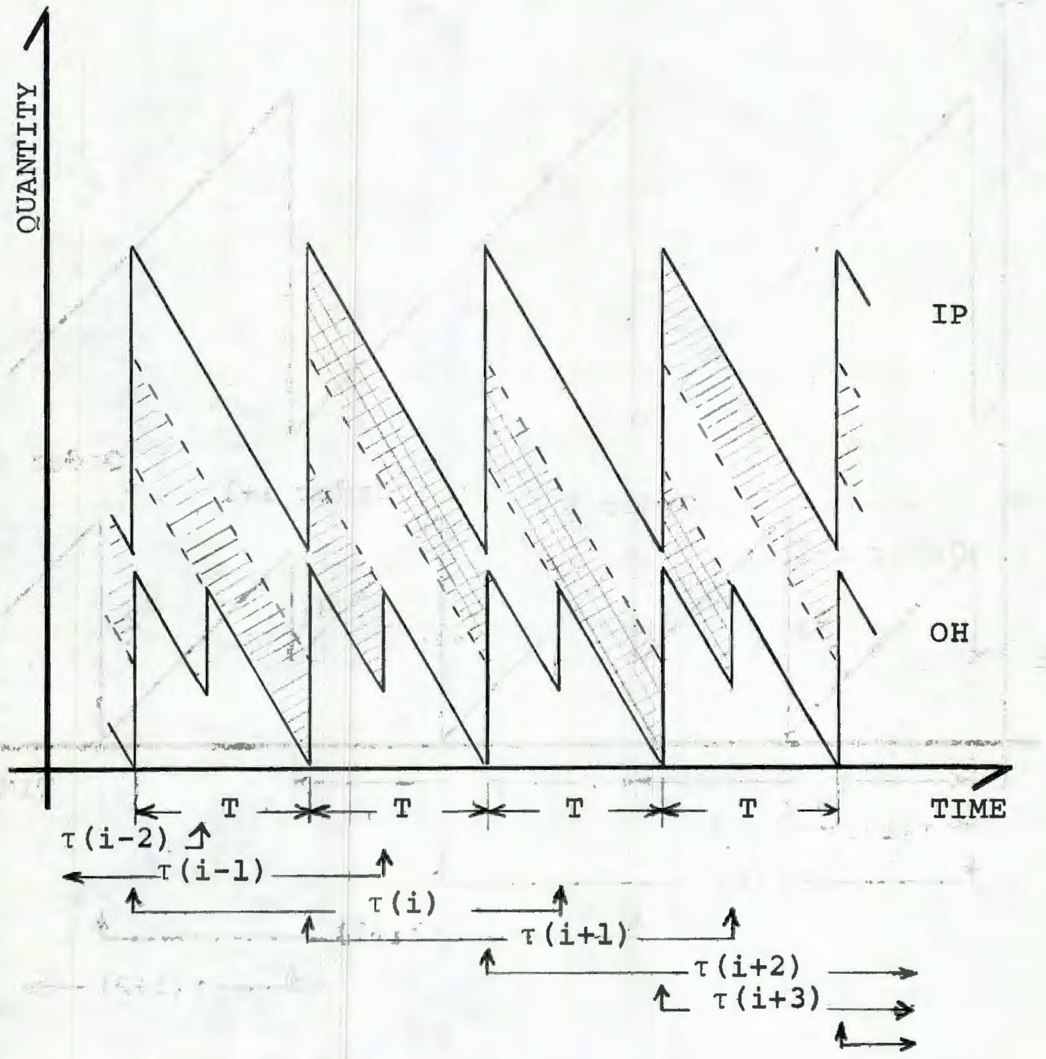


FIGURE 9B

TWO INCREMENT MODEL

$2T_w < \tau < 3T_w$

that $K^* = K_w - ICf'\lambda\tau'$ will be the proper minimal cost. It remains to define b*. It is obvious that the same technique of using $\tau''$ in lieu of $\tau$ gives the proper optimal values for b*.

Suppose now, that shortages are allowed and there is a cost $\pi$ associated with the occurrence of each shortage and a shortage cost rate $\hat{\pi}$ per shortage per unit time. In the lost sales case, $\hat{\pi}$ must be equal to zero since it is meaningless to discuss the duration of a lost sale. The cost $\pi$ may be interpreted in this case as the profit lost as a result of not making the sale. Both $\pi$ and $\hat{\pi}$ may be applicable to the backorder case. The term $\pi$ is interpreted here as the cost of processing a backorder which is independent of the time for which the backorder is outstanding. The $\hat{\pi}$ term is the portion of the backorder cost which is dependent upon the duration of the backorder.

Some preliminary calculation can be made on the effect of shortages on holding cost and of the direct shortage costs prior to modifying the formula for total cost, K. Define s as the maximum number of shortages occurring at one time. By the previous arguments reducing the problem to one case, we will assume that $f_1 \geq \tau/T$. There are two possibilities for shortages. They may occur prior to one or both deliveries. Figure 10A illustrates the case where shortages occur only once during a cycle, at the end, since $f_1 \geq \tau/T$. The number of shortages per cycle is s and the average number of shortages per unit time is $\frac{1}{2}$ s $(s/\lambda)$. Therefore shortage cost per unit time is

$$\frac{\pi\lambda s}{Q} + \frac{s^2\hat{\pi}}{2Q} .$$

Holding costs per unit time must be reduced by

$$IC\left(s-\frac{s^2}{2Q}\right) .$$

Figure 10B illustrates the case where shortages occur prior to both deliveries. The number of shortages per cycle is $2s - (q_1-\lambda\tau)$. As expected, if $q_1 = \lambda\tau$, the number of shortages per cycle is 2s. This term is multiplied by $\pi$ and divided by T, to get the shortage cost factor to add to K. In a similar fashion, the applicable cost factor involving $\hat{\pi}$ is found to be

$$\frac{\hat{\pi}}{2Q}\left[s^2 + (s-(f_1Q-\lambda\tau))^2\right] .$$

The holding cost adjustment is:

$$IC\left[\frac{s^2 + (s-(f_1Q-\lambda\tau))^2-s}{2Q}\right] .$$

Now there are two cases to be considered.

Case I $q_1 \geq \lambda\tau$ and shortages only at end of cycle.

Case II $q_1 \geq \lambda\tau$ and shortages before each delivery.

These cases can be solved by adding the appropriate factors above to equation (4.1). It is also convenient to substitute $f_1Q$ for $q_1$ to make obvious the proper functional dependence upon Q.

It is no longer necessary to include b in the equations as a decision variable, since s serves the purpose for calculation and optimization and the value of $b^* = -s^*$. The

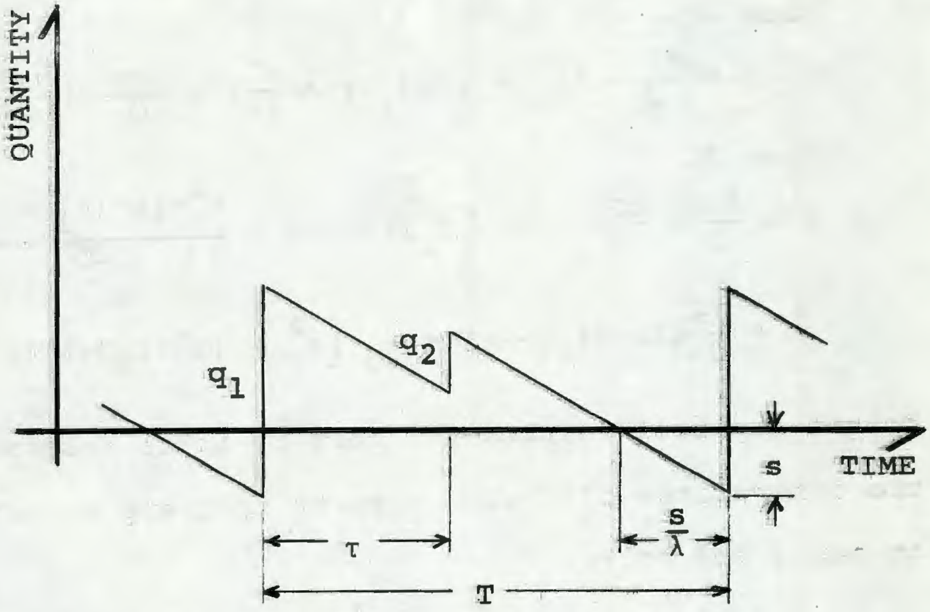


FIGURE 10A

TWO INCREMENT MODEL WITH SHORTAGES
AT END OF CYCLE ONLY

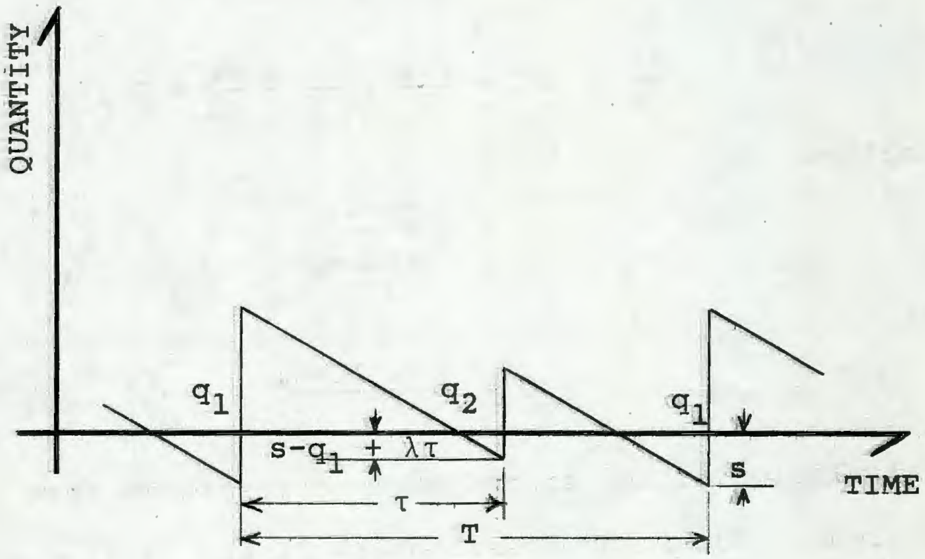


FIGURE 10B

TWO INCREMENT MODEL WITH SHORTAGES
BEFORE EACH DELIVERY

equations for K are given below.

Case I

$$K = \frac{A\lambda}{Q} + \frac{ICQ}{2} + IC(f_1\lambda\tau - s + \frac{s^2}{2Q}) + \frac{\lambda\pi s}{Q} + \frac{\hat{\pi}s^2}{2Q} .$$

Case II

$$K = \frac{A\lambda}{Q} + \frac{ICQ}{2} + IC\left[f_1\lambda\tau - \lambda\tau - s + \frac{s^2 + (s-(f_1Q-\lambda\tau))^2}{2Q}\right]$$

$$+ \frac{\lambda\pi}{Q}(2s-(f_1Q-\lambda\tau) + \frac{\hat{\pi}}{2Q}\left[s^2 + (s-(f_1Q-\lambda\tau))^2\right] .$$

To get explicit results for case I, it is necessary to take the derivatives of K with respect to Q and s, set them equal to zero, and solve.

$$\frac{\partial K}{\partial Q} = \frac{-A\lambda}{Q^2} + \frac{IC}{2} - \frac{ICs^2}{2Q^2} - \frac{\lambda\pi s}{Q^2} - \frac{\hat{\pi}s^2}{2Q^2} = 0 ,$$

implies

$$Q = \left[\frac{2\lambda(A+s\pi) + s^2(\hat{\pi}+IC)}{IC}\right]^{\frac{1}{2}} . \tag{4.2}$$

$$\frac{\partial K}{\partial s} = -IC + \frac{ICs}{Q} + \frac{\lambda\pi}{Q} + \frac{\hat{\pi}s}{Q} = 0 ,$$

implies

$$s = \frac{QIC-\lambda\pi}{(IC+\hat{\pi})} .$$

or

$$Q = \frac{\lambda\pi+s(IC+\hat{\pi})}{IC} \tag{4.3}$$

Calculation of the second derivatives shows that we have a minimum. Combining equations (4.2) and (4.3) yields the following quadratic in s;

$$(\hat{\pi}^2 + \hat{\pi}IC)s^2 + 2\pi\hat{\pi}\lambda s = 2\lambda AIC - (\pi\lambda)^2 .$$

Note that if $\hat{\pi} = 0$ then either there is no solution for $0 < s < \infty$ or if $(\lambda\pi)^2 = 2\lambda AIC$ any value of s is a solution.

In words, this implies that if $\hat{\pi} = 0$ and $\pi \neq 0$, then either there should be no shortages or infinitely many. That is, the system should incur no shortages or else go out of business. The lost sales case is characterized by just these conditions $\hat{\pi} = 0$ and $\pi \neq 0$. If $\hat{\pi} \neq 0$ then

$$Q^* = \left[\frac{2\lambda A(\hat{\pi}+IC)-(\lambda\pi)^2}{\hat{\pi}IC}\right]^{\frac{1}{2}} ,$$

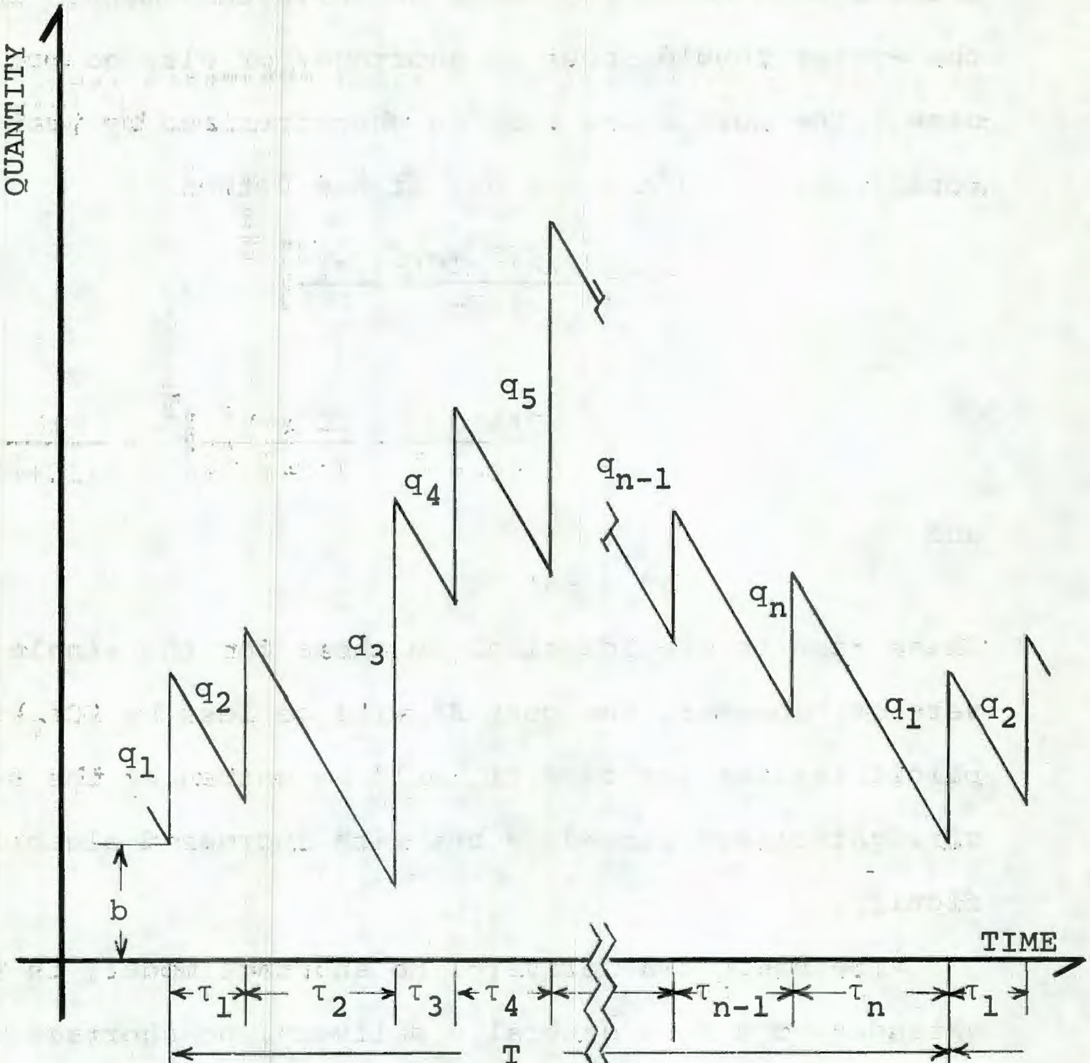
$$s^* = \left[\frac{2A\lambda IC}{(IC+\hat{\pi})\hat{\pi}} - \frac{IC(\lambda\pi)^2}{(IC+\hat{\pi})^2\hat{\pi}}\right]^{\frac{1}{2}} - \frac{\pi\lambda}{(IC+\hat{\pi})} ,$$

and

$$b^* = -s^* .$$

These results are identical to those for the single increment version, however, the cost $K^*$ will be less by $ICf_2\lambda\tau$. Explicit results for case II could be gotten by the same straightforward procedure but with increased algebraic difficulty.

The basic two delivery, no shortage model, is readily extended to a more general n delivery, no shortage model. Figure 11 illustrates such a model. Modified notation is required. Assume n fractional deliveries $f_1, f_2, \ldots, f_n$ where $\sum_{i=1}^{n} f_i = 1$. Then $f_iQ = q_i$ and there are n deliveries of size $q_1, q_2, \ldots q_n$ where $\sum_{i=1}^{n} q_i = Q$. $\tau_i$ is defined for $i=1,2,\ldots,n-1$ to be the time between the ith and i+1st delivery. $\sum_{i=1}^{n-1} \tau_i \leq T$ for a feasible solution to exist. It is obvious by comparison with the basic model, that $Q^* = Q_w$.



NOTE: $\tau_n = T - \sum_{i=1}^{n-1} \tau_i$

FIGURE 11

MULTI-INCREMENT MODEL

It is only a bit more difficult to conclude from the definition of b and the geometry of the model that

$$b^* = \text{MAXIMUM}\left(0, \lambda\tau_1 - q_1, \lambda(\tau_1+\tau_2)-(q_1+q_2), \ldots, \sum_{j=1}^{i} \lambda\tau_j - q_j, \ldots \right.$$

$$\left. \sum_{j=1}^{n-1} \lambda\tau_j - q_j, \sum_{j=1}^{n} \lambda\tau_j - Q\right) .$$

Similarly an expression for K* can be found.

$$K^* = K_w - IC\left[ b^* - \sum_{i=2}^{n} \left( q_i \sum_{j=1}^{i-1} \tau_j \right) \right] .$$

No attempt is made to consider the n increment model with shortages allowed here, for several reasons. First, the formulation involves too many different cases to be of much interest unless severe and unrealistic assumptions are made. In fact, there are 2n-1 different possibilities for distinct cases, since the on hand curve has 2n distinct vertices and the zero line may fall optimally between any pair of them. The solution procedure would, of necessity, follow the technique of assuming a particular case and determining if the optimal values are consistent with that assumption. After all consistent solutions are found, it is then necessary to compute and compare the cost K resulting from each. The minimum K is K* and the associated optimal values are the optimal policy. This is a tedious technique, but is usually susceptible to computer solution for any specific application. The second reason for not pursuing this further is that it is doubtful if the model would have any significant applications for a value of n greater than two. It seems far more likely that if the number of deliveries is large, their size and the time between deliveries would be uniform, or at least subject to small random variation. In this event, other models presented provide a close approximation and are much easier to use.

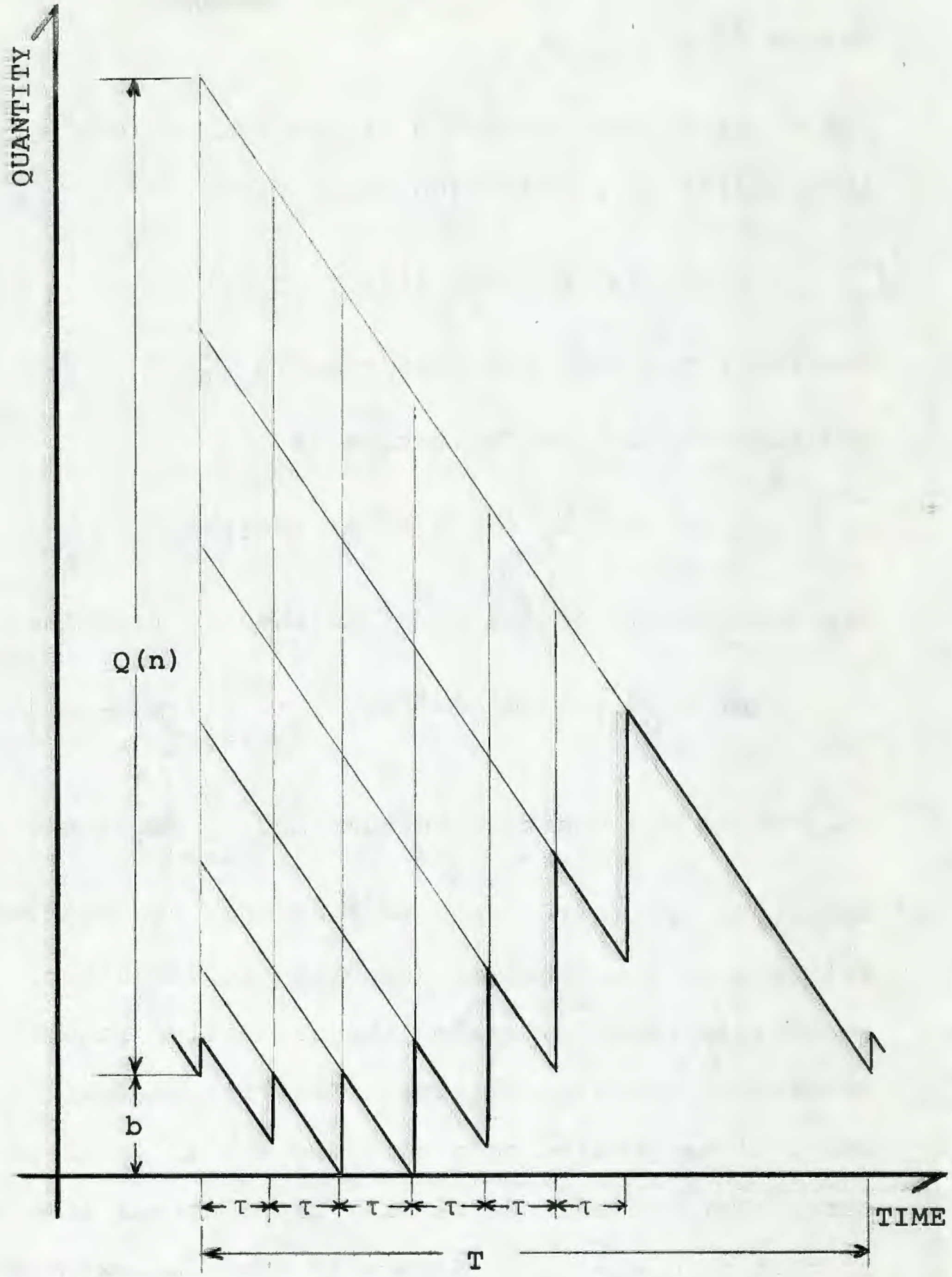
One particular delivery pattern that might warrant further analysis at a later date is the possibility that either the first or last of a large number of shipments might be an odd size, while all of the other lots are the same size. This could occur, for example, as the result of the source having a partial quantity on hand for immediate delivery and producing the remainder and shipping in equal increments. It appears probable that significant simplifications could be made.

### 4.3 Increasing Delivery Lot Size at Equal Intervals

It is assumed in this model that a manufacturer with an increasing production rate has agreed to make shipments at fixed intervals, of the entire production since the last shipment. Let $\tau$ be the fixed interval and $p_i$ the production in the ith period. $p_i > p_j$ if and only if $i > j$ since we assume an increasing production rate. The usage during each period is $\lambda\tau$ and the only parameter over which the manager has control is n. No shortages are allowed. See Figure 12. Now there exists an i such that $p_i \geq \lambda\tau$,

and there exists a j such that $\sum_{i=1}^{j} p_i \geq j\lambda\tau$,

therefore the system has a feasible solution.



NOTE: a = 1, a' = 1, n = 7, $\lambda\tau = 3$

FIGURE 12

INCREASING PRODUCTION RATE:
DELIVERY AT FIXED INTERVALS

Define $Q(n) = \sum_{i=1}^{n} p_i$ .

The on hand level required at the time of the first delivery in a series to prevent shortages is

$$b = \max_{i} \left(i\lambda\tau - \sum_{j=1}^{i} p_j\right) .$$

Now the order cost per unit time is $\frac{A\lambda}{Q(n)}$ ,

and holding cost per unit time is

$$IC\left[\frac{Q(n)}{2} + b - \sum_{i=2}^{n} (i-1)\tau p_i\right].$$

The formulation of the model is then to minimize

$$K(n) = \frac{A\lambda}{Q(n)} + IC\left[\frac{Q(n)}{2} + b - \tau \sum_{i=2}^{n} ip_i - p_i\right]$$

subject to n a positive integer and $\sum_{i=1}^{n} p_i \geq n\lambda\tau$. Specific solutions are fairly easy to find when the production quantities have a convenient algebraic relationship. As a reasonable example, assume the production rate is a linear increasing function of time. That is, assume $\Psi = a + a't$ where a' is greater than zero and a + a' is greater than zero. For convenience, we will use $\tau$ as our time interval so that $p_i = a + ia'$. Figure 12 shows an example of the on hand value resulting from this delivery pattern. The decision variable is n, the only parameter under our control. The total order quantity Q is a function of n denoted Q(n) where

$$Q(n) = a + a' + a + 2a' + \dots + a + ia' + \dots + a + na'$$

$$= na + a' \sum_{i=1}^{n} i = na + \frac{a'n(n+1)}{2} .$$

To prevent shortages, a quantity b must be on hand at the time of first delivery. The on hand decreases at each successive low point, until $a + ia' \geq \lambda\tau$. Let $y = i - 1$. Then y is an integer and

$$b = \lambda\tau y - ya - \frac{a'y(y+1)}{2} .$$

It is necessary that the order quantity Q(n) be greater or equal to the demand during the time over which it is delivered. That is,

$$Q(n) \geq n\lambda\tau ,$$

or

$$a + \frac{a'(n+1)}{2} \geq \lambda\tau ,$$

which can be rewritten

$$n \geq \frac{2\lambda\tau - a - a'}{a'} .$$

We know the order cost per cycle is A, so that the order cost per unit time is

$$\frac{A\lambda}{Q(n)} = \frac{2A\lambda}{n(2a+a'n+a')} .$$

The holding cost per cycle can be seen from Figure 12 using previously described techniques to be

$$IC\left[\tfrac{1}{2}Q(n)T - \Big(\tau(a+2a') + 2\tau(a+3a') + \dots + i\tau(a+a'+ia') + \dots + (n-1)\tau(a+a'+(n-1)a')\Big) + bT\right] ,$$

so that holding cost per unit time is

$$\frac{IC}{2} Q(n) + ICb - \frac{IC\tau 2\lambda}{n(2a+na'+a')}\left[\sum_{i=1}^{n-1} i(a+a') + \sum_{i=1}^{n-1} i^2 a'\right].$$

ICb is independent of n and can be dropped. We can now combine these terms to get total variable cost

$$K = \frac{2A\lambda}{n(2a+a'n+a')} + \frac{ICan}{2} + \frac{ICa'(n^2+n)}{4} -$$

$$\frac{IC\lambda\tau}{3}\left[\frac{2a'n^2+3an+6a'n-3a-2a'}{2a+na'+a'}\right].$$

Setting the derivative of K with respect to n equal to zero, yields a 5th degree polynomial with exactly one sign change which by Descarte's Rule implies exactly one positive root. The second derivative is positive, so there exists a unique $n > 0$ such that K is a minimum. There is no algebraic exact method for computing the root of a 5th degree polynomial, so the Newton-Raphson method or some other convergent iteration technique must be used to find this value of n. Call this value n'. If

$$n' \geq \frac{2\lambda\tau-a-a'}{a'}$$

it is feasible and therefore the desired solution is $n^* = n'$. If

$$n' < \frac{2\lambda\tau-a-a'}{a'} ,$$

then n* is the smallest integer larger than or equal to

$$\frac{2\lambda\tau-a-a'}{a'} ,$$

since K(n) increases monotonically with n for all $n > n'$.

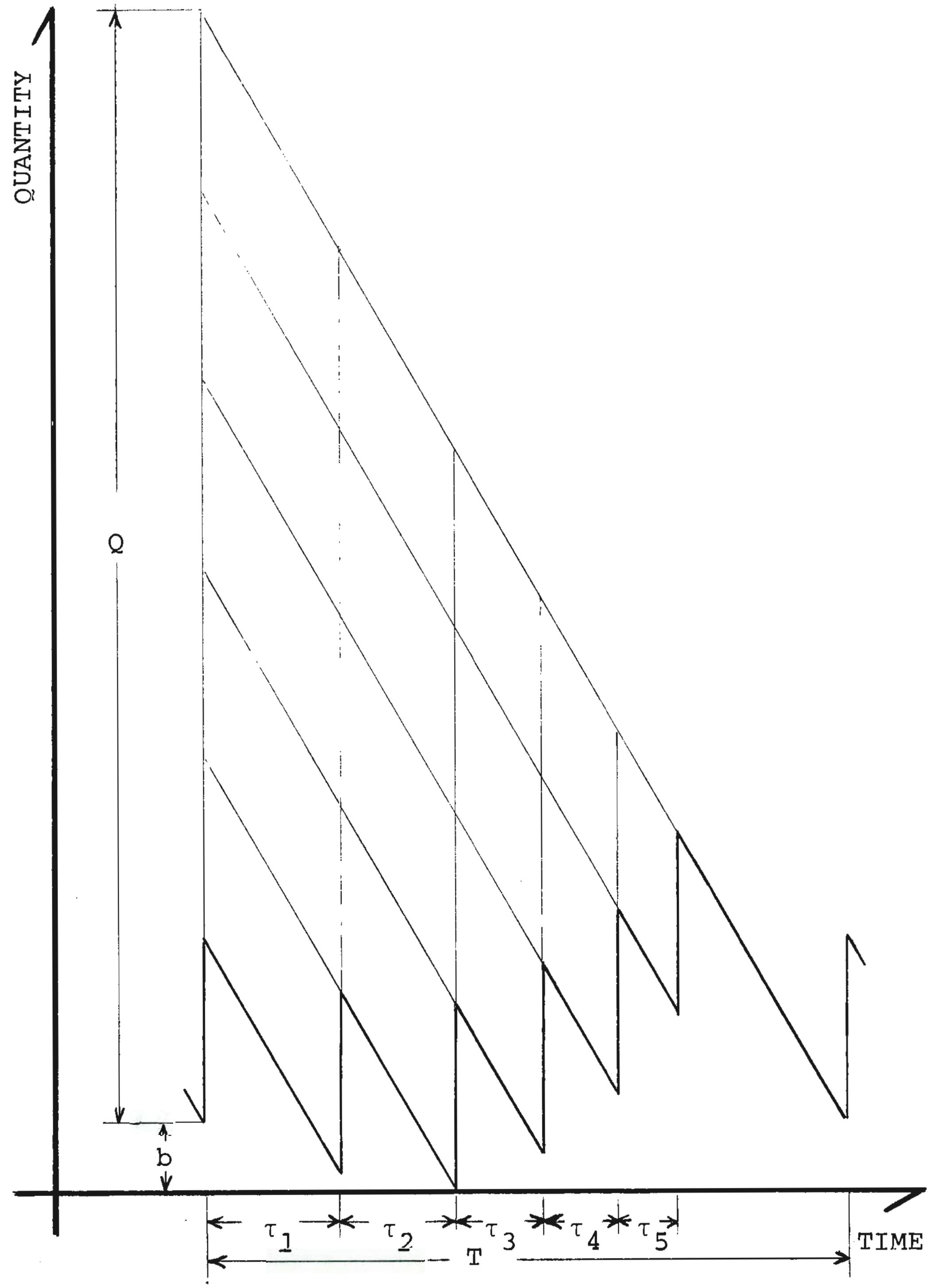
## 4.4 Equal Delivery Lots at Decreasing Intervals

In this section, it is assumed that a producer has an increasing production rate and will make a shipment of a fixed size at unequal intervals. For clarification, an example is given. United Widget makes gismos for our company. When we contact them, we are informed that they have a government contract in production now, but that they can produce 100 per month for us now and will add a 100 per month production line each month for 8 months as the government contract expires. Gismos are much cheaper to ship in carload lots of 200, so he offers to ship each carload lot of 200 as soon as it is completed. We would expect shipments at approximately 1.5 months, 2.3 months, 3 months, 3.5 months, etc.

The general problem can be formulated as follows. We get a quantity q at intervals $\tau_i$ where the increasing rate implies that $\tau_i > \tau_j$ if and only if $i < j$. Our decision variable is n, and the total order quantity will be nq. For feasibility $nq \geq \lambda \sum_{i=0}^{n-1} \tau_i$ where $\tau_0$ is the delay until the first shipment is received. Figure 13 shows an example of a cycle of this model. It is obvious using the, by now familiar, techniques that

$$K(n) = \frac{A\lambda}{nq} + \frac{ICnq}{2} - ICq\left[\sum_{i=1}^{n-1} i\tau_{n-1}\right].$$

For convenience, we will use difference equations rather than taking derivatives. That is, define



NOTE: Data same as text.

FIGURE 13

INCREASING PRODUCTION RATE:
FIXED DELIVERY QUANTITIES

$\Delta K = K(n+1) - K(n)$. Our optimal integer value of n is the smallest feasible value of n such that $\Delta K$ is positive. Find the smallest n such that

$$\Delta K = \frac{-A\lambda}{q(n^2+n)} + \frac{ICq}{2} - \frac{IC\lambda}{n} \sum_{i=1}^{n} \tau_i > 0 \ .$$

This can be rewritten as

$$(n^2+n) \left[ 1 - \frac{\lambda}{nq} \sum_{i=1}^{n} \tau_i \right] > \frac{2A\lambda}{q^2 IC} \ .$$

It is clear that the left side of the inequality is a monotonically increasing function of n since the $\tau_i$ are decreasing. Now we call the smallest integer value of n that satisfies this inequality n'. For feasibility we must have

$$n \geq \frac{\lambda}{q} \sum_{i=0}^{n-1} \tau_i \ ,$$

so that if

$$n' \geq \frac{\lambda}{q} \sum_{i=0}^{n'-1} \tau_i \ ,$$

then $n^* = n'$.

If

$$n' < \frac{\lambda}{q} \sum_{i=0}^{n'-1} \tau_i \ ,$$

then $n^*$ is the smallest integer value of n such that

$$n \geq \frac{\lambda}{q} \sum_{i=0}^{n-1} \tau_i \ .$$

# CHAPTER 5

# SPECIAL CONSIDERATIONS IN PHASED DELIVERY MODELS

## 5.1 Shortages

The problem of incorporating shortages and shortage costs into incremental delivery models can be exceedingly complex or, in special cases, quite simple. There are several techniques for approximation which may be nearly exact under certain conditions. In this section, a technique for exact solution of deterministic models with shortages will be described and three approximation techniques will be presented. Reference to earlier models and simple examples will be given as appropriate.

Figure 14 shows a typical graph of on hand level delivery for a cycle of n incremental deliveries, resulting from a single contract. It is obvious that the on hand level can be lowered or raised in relation to the zero stock level to balance holding and shortage costs with each other and with order costs to achieve a minimum total cost. The problem in formulating an exact model is based on the fact that there are 2n - 1 different forms for the shortage cost, depending upon where the zero line cuts the on hand curve. There are 2n points connected by the on hand curve and the zero line may lie above 1, 2, .... 2n - 2, or 2n - 1 of them. Call these levels 1, ... , 2n - 2 and 2n - 1. It will never be optimal for the on hand curve to lie entirely above or below the zero line if infinite and zero

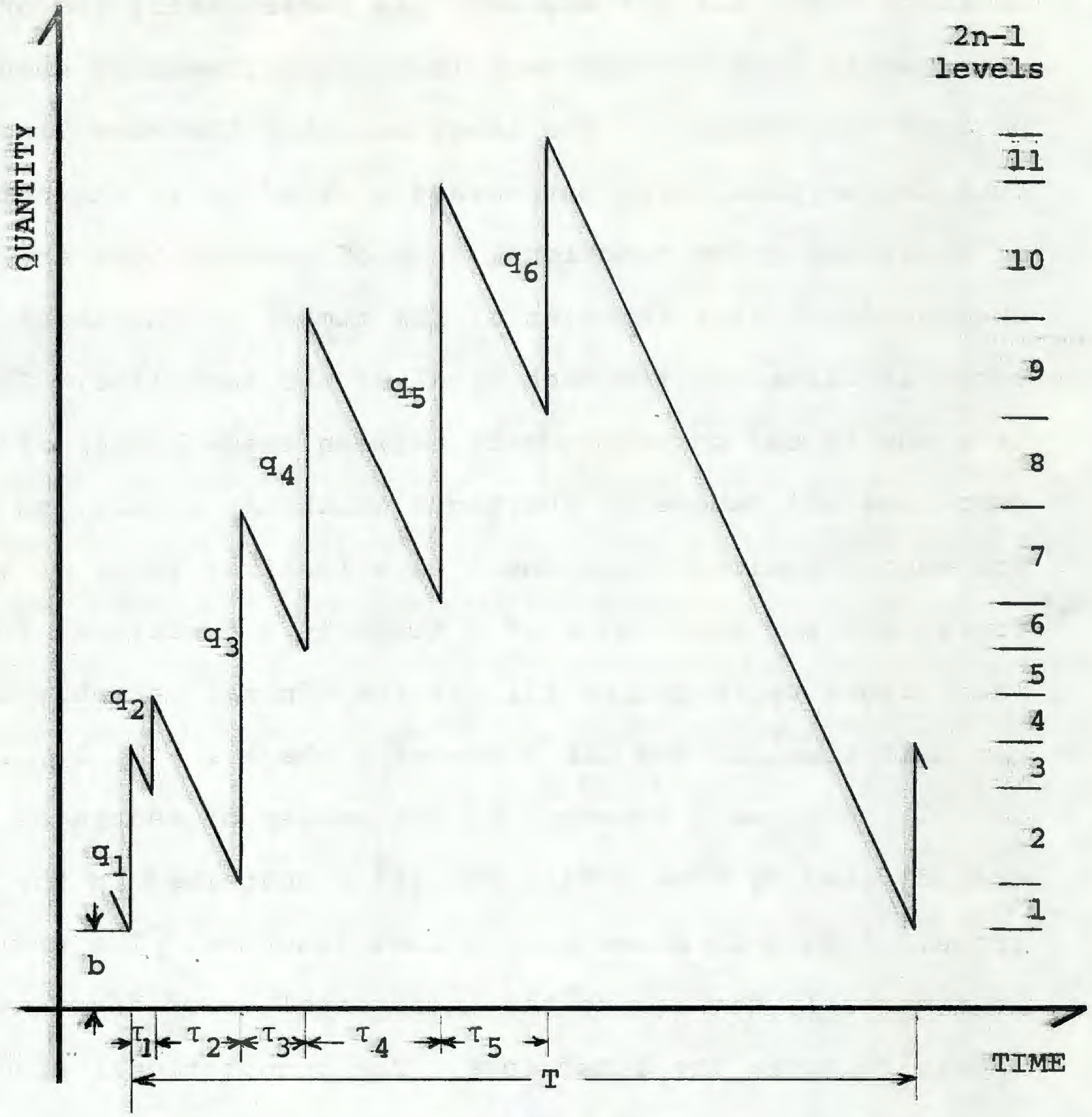


FIGURE 14

INCREMENTAL DELIVERIES WITH SHORTAGE LEVELS shortage costs are not allowed. In these cases, the on hand curve would "sit on" the zero line or no inventory should be kept respectively. The level at which the zero line cuts the on hand curve determines a value of s, the number of shortages. The functional form of holding cost and shortage cost as a function of the number of shortages per cycle is different for each level of the zero line. There is a one to one correspondence between these levels of the zero line and ranges of shortages which may result, so that for each functional form there is a feasible range of values for s, and for each value of s there is a functional form. This allows us to define K(i) as the minimal variable cost per unit time for the ith range of s where i = 1, 2 ... , 2n - 1. K(s) as a function of the number of shortages is then defined by K(s) = K(i) for all s contained in the ith range. Figure 15 shows such a cost function. The problem is remarkably similar to the incremental price discount models found in the literature. The computational procedure is virtually identical. First, compute the optimal policy $p_i$ for each K(i) i = 1, ..., 2n - 1. The policy $p_i$ includes a value for shortages which may or may not lie within the allowable range for s. For each of the policies $p_i$ for which $s_i$ is included in the ith range for s, compute $K(i,p_i)$. The optimal policy p* is then the $p_i$ for which $K(i,p_i)$ is minimal.

Less it sound overly simple to solve such a problem, recall that the 2n - 1 functional forms may be very complex

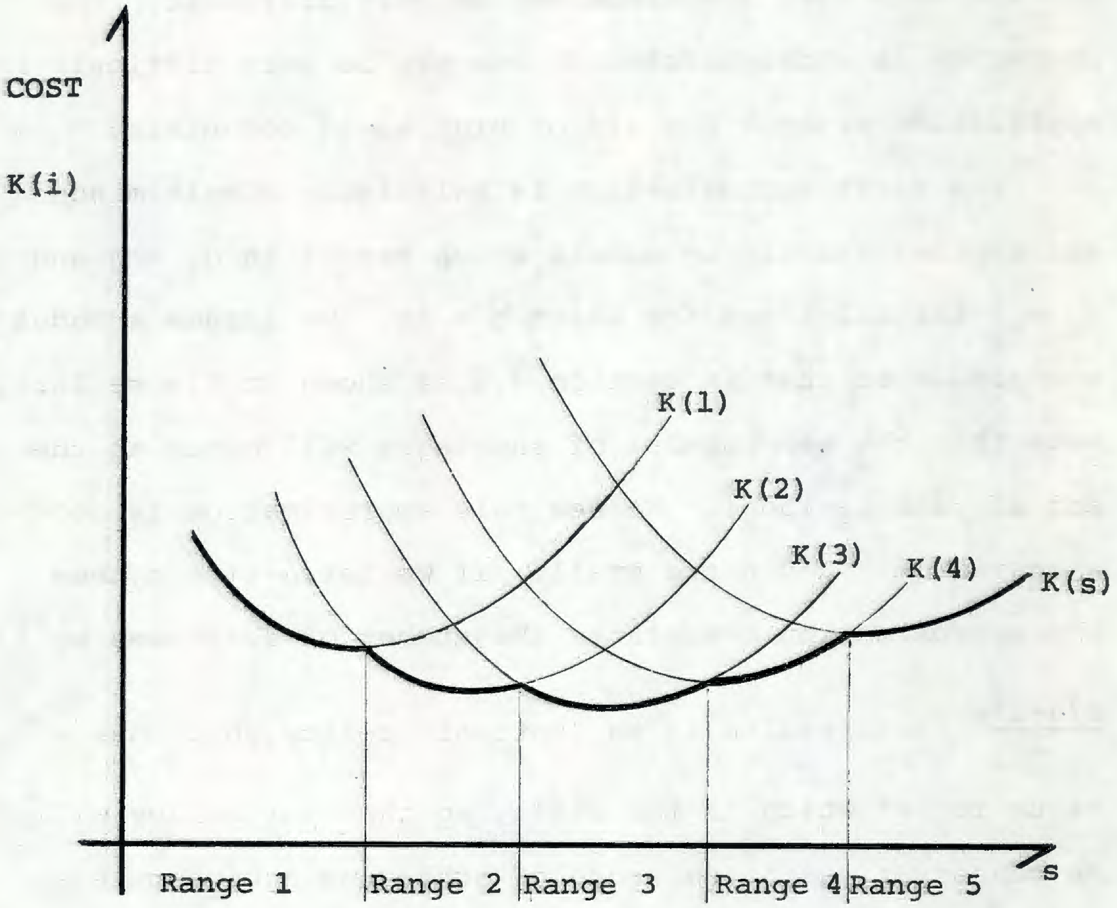


FIGURE 15

COST FUNCTION FOR K(i)

and disparate if the $\tau_i$ and $q_i$ are unequal. The zero line may cut the on hand curve at 1, 3, 5, 7 ... different points and the problem of specifically defining the holding cost and shortage cost functions may be very difficult. The procedure is straightforward, but may be very difficult in application without the aid of high speed computers.

The first approximation is relatively simple minded and applies exactly to models which result in $q_i = q$ and $\tau_i = \tau$ for all i and for which $q = \lambda\tau$. We assume a model comparable to that in section 3.4 as shown in Figure 16. Note that the same number of shortages will occur at the end of each period $\tau$. We see this approximation is good whenever $q-\lambda\tau$ and n are small. If we let $q-\lambda\tau = e$, then the approximation overstates the number of shortages by $\frac{n(n-1)e}{2}$ and results in an "optimal" policy which has a value for s* which is too small, so that our policy will be conservative in the sense of achieving a higher level of service than forecast. It is interesting to calculate specific results for the shortage version of the model in section 3.4. Recall that $q = \lambda\tau$ will be optimal.

Assume the source of an item agrees to produce at a rate equal to $\lambda$, the demand rate. The values of A, B, C, I and $\lambda$ are known. Shortages will be backordered and the cost per unit per unit time of a backorder is $\hat{\pi}$. The cost per backorder independent of time is $\pi$. Our decision variables are q, n and s, the number of shortages prior to

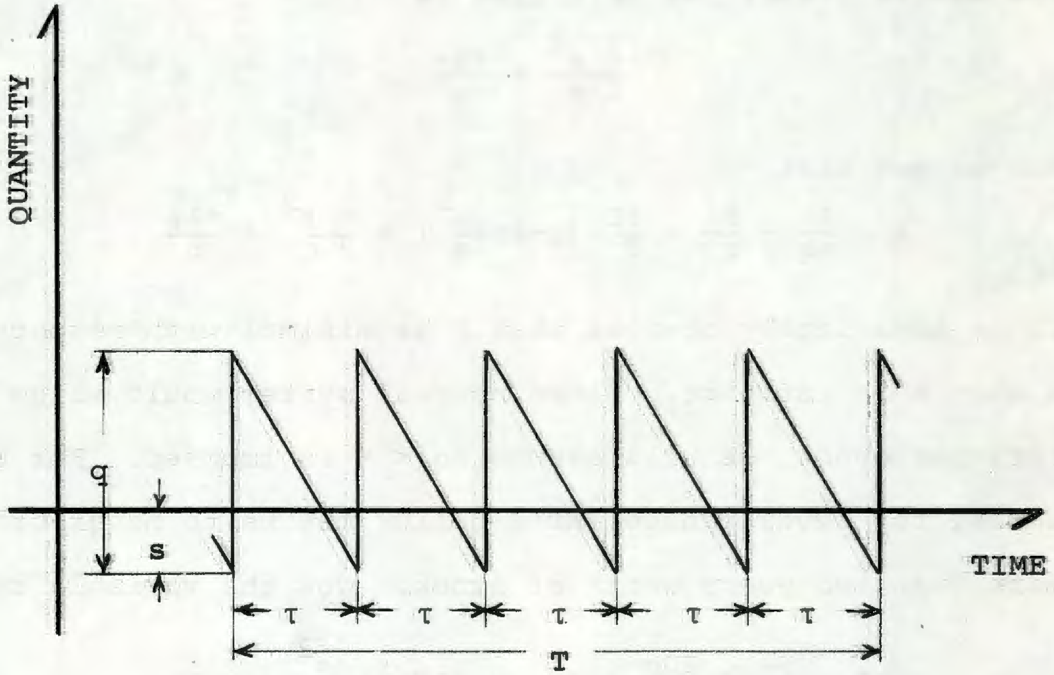


FIGURE 16

SHORTAGE MODEL

APPROXIMATION ONE

receipt of a lot. Now order cost per unit time is

$$\frac{A\lambda}{nq} + \frac{B\lambda}{q} ,$$

holding cost per unit time is

$$\frac{IC}{2} \frac{(q-s)^2}{q} ,$$

and shortage cost per unit time is

$$\frac{\hat{\pi}}{2} \frac{s^2}{q} + \frac{\pi\lambda s}{q} .$$

Now we see that

$$K = \frac{A\lambda}{nq} + \frac{B\lambda}{q} + \frac{IC}{2} (q-2s+\frac{s^2}{q}) + \frac{\hat{\pi}}{2} \frac{s^2}{q} + \frac{\pi\lambda s}{q} .$$

It is immediately obvious that K is minimal with respect to n when n is infinite. Since no real system would allow an infinite order, we will assume $nq < M$ is imposed. For example, top level management's policy may be to never order more than two years worth of stock. Now the variable cost is

$$K = \frac{B\lambda}{q} + \frac{ICq}{2} - ICs + (IC+\hat{\pi}) \frac{s^2}{2q} + \frac{\pi\lambda s}{q} .$$

The optimal value of n is

$$n^* = \frac{M}{q^*} .$$

Using the usual technique, the derivatives with respect to q and s yield

$$q = \left[\frac{2B\lambda}{IC} + \frac{(IC+\hat{\pi})s}{IC} + \frac{2\pi\lambda s}{IC}\right]^{\frac{1}{2}} ,$$

and

$$s = \frac{qIC-\pi\lambda}{IC+\hat{\pi}}$$

respectively. The second equation can be rewritten as

$$q = \frac{(IC+\hat{\pi})s}{IC} + \frac{\pi\lambda}{IC} .$$

It is interesting to note that when $\pi = 0$ the ratio of the time stock is on hand to the time stock is out equals

$$\frac{(q-s)\lambda}{s\lambda} = \frac{\hat{\pi}}{IC} ,$$

the ratio of the shortage and holding cost rates. Setting the two equations for q in terms of s equal and rearranging we get

$$(\hat{\pi}^2 + \hat{\pi}IC)s^2 + 2\pi\hat{\pi}\lambda s + (\pi\lambda)^2 - 2B\lambda IC = 0 .$$

It is obvious that if $\hat{\pi} = 0$ and

$$(\pi\lambda)^2 \neq 2B\lambda IC ,$$

then $s = 0$ or $\infty$, and it is optimal to either have no shortages or to go out of business. So we require $\hat{\pi} > 0$ and solve for the optimal values of q, s, and n. We find

$$q^* = \left[\left(\frac{IC+\hat{\pi}}{\hat{\pi}}\right)\left(\frac{2B\lambda}{IC} - \frac{(\pi\lambda)^2}{IC(\hat{\pi}+IC)}\right)\right]^{\frac{1}{2}} ,$$

$$s^* = \frac{q^*IC-\pi\lambda}{(\hat{\pi}+IC)} ,$$

and

$$n^* = M/q^* .$$

These results are very closely related to the single lot model's results. The optimal values for s and q are identical, except that the lot order cost, B, is used as an order instead of A. As we have seen before, we can use the standard formulas and then order a sufficient number of increments of the prescribed size to make a total order of M.

The second approximation technique assumes that the cost of shortages is high, relative to the other costs of the model, particularly the holding cost, so that shortages will be rare. Shortages are assumed to occur only prior to the first delivery in a cycle. Specifically, it is assumed that $q_1 \geq \lambda\tau_1 + s$ and the approximation will be exact if, in fact, this condition holds for the optimal values. That is, if $q^* \geq \lambda\tau^* + s^*$. This approximation cannot be applied successfully when we have full control over delivery conditions since then $q^* = \lambda\tau^*$. Reference to Figure 17 will assist in recognizing that this approximation results in under stating shortages in the model. It is not a conservative policy, since the actual number of shortages and duration of shortages will exceed the model forecast and result in reduced service. For example, a shortage model similar to the model of section 3.2 where q and $\tau$ are fixed would be approximated by

$$K = \frac{A\lambda}{nq} + \frac{B\lambda}{q} + \frac{IC}{2}(nq-n\lambda\tau+\lambda\tau-2s + \frac{s^2}{nq}) + \frac{\pi s\lambda}{nq} + \frac{\hat{\pi}s^2}{2nq} .$$

The only decision variables are n and s since q and $\tau$ are fixed.

$$\frac{\partial K}{\partial n} = \frac{-A\lambda}{n^2q} + \frac{IC}{2}(q-\lambda\tau) + \frac{IC}{2}\frac{s^2}{n^2q} - \frac{\pi s\lambda}{n^2q} - \frac{\hat{\pi}s^2}{2n^2q} = 0 ,$$

and

$$\frac{\partial K}{\partial s} = -IC + \frac{ICs}{nq} + \frac{\pi\lambda}{nq} + \frac{\hat{\pi}s}{nq} = 0 ,$$

imply

$$s^* = \frac{ICnq-\pi\lambda}{IC+\hat{\pi}} ,$$

and

$$n^* = \left[\frac{2A\lambda+2\pi\lambda s+(IC+\hat{\pi})s^2}{q(q-\lambda\tau)}\right]^{\frac{1}{2}} .$$

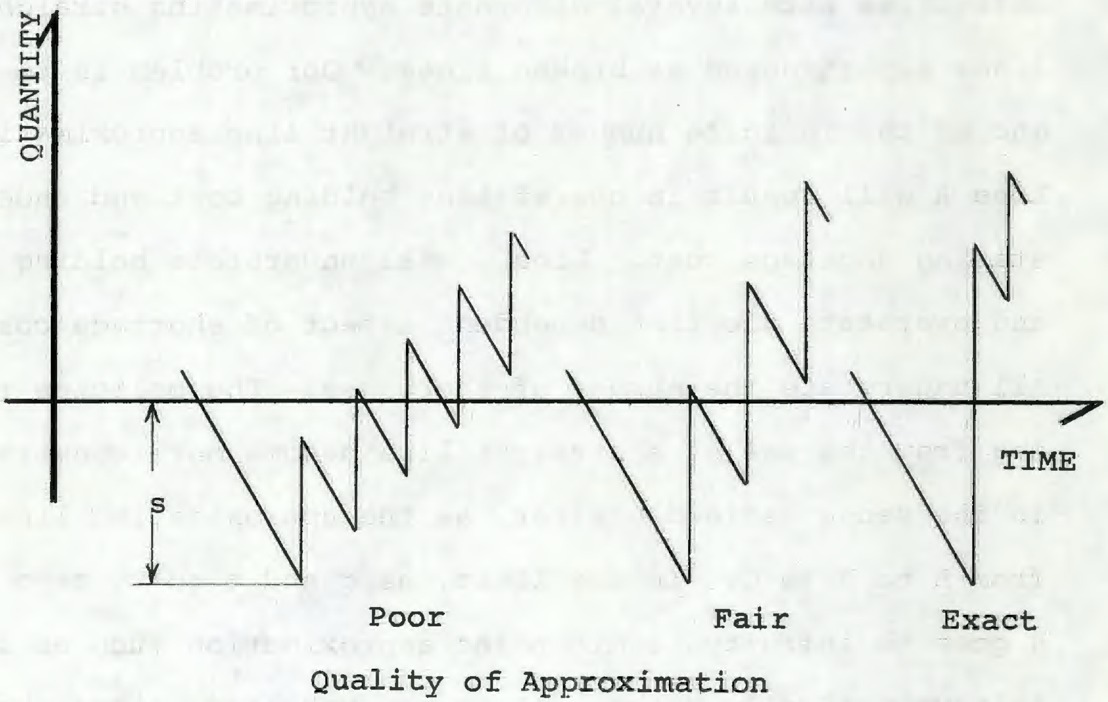


FIGURE 17

SHORTAGE MODEL

APPROXIMATION TWO

The final approximation technique to be presented here is designed to fit systems where $q > \lambda\tau$ and both $q$ and $\tau$ are small while $n$ is large. It is basically an approximation which reduces the "saw tooth" form of a rising on hand curve to a straight line,as in the continuous finite production rate model. Figure 18 shows a typical cycle of incremental deliveries with several alternate approximating straight lines superimposed as broken lines. Our problem is to choose one of the infinite number of straight line approximations. Line A will result in overstating holding cost and understating shortage cost. Line C will understate holding cost and overstate the time dependent aspect of shortage cost. All understate the number of shortages. The policies resulting from the use of a straight line become more conservative, in the sense defined earlier, as the approximating line moves from A to B to C. In the limit, as $q$ and $\tau$ go to zero while $n$ goes to infinity, a mid-point approximation such as line B is asymptotically exact. It is not unbiased, since shortages are always understated. On the other hand, line C has the advantage of being the most conservative approach and also provides the "best" forecast of shortage costs, since it understates the multiplier of $\pi$, while overstating the multiplier of $\hat{\pi}$. We can wait until after the calculations are made to decide which approximation to use.

Figure 19 shows a single cycle of the approximating model. The slope of the rising on hand curve is $q/\tau - \lambda$. Recall that if either $q$ or $\tau$ are variable, then $q = \lambda\tau$ and this model will not apply, so our decision variables are $n$

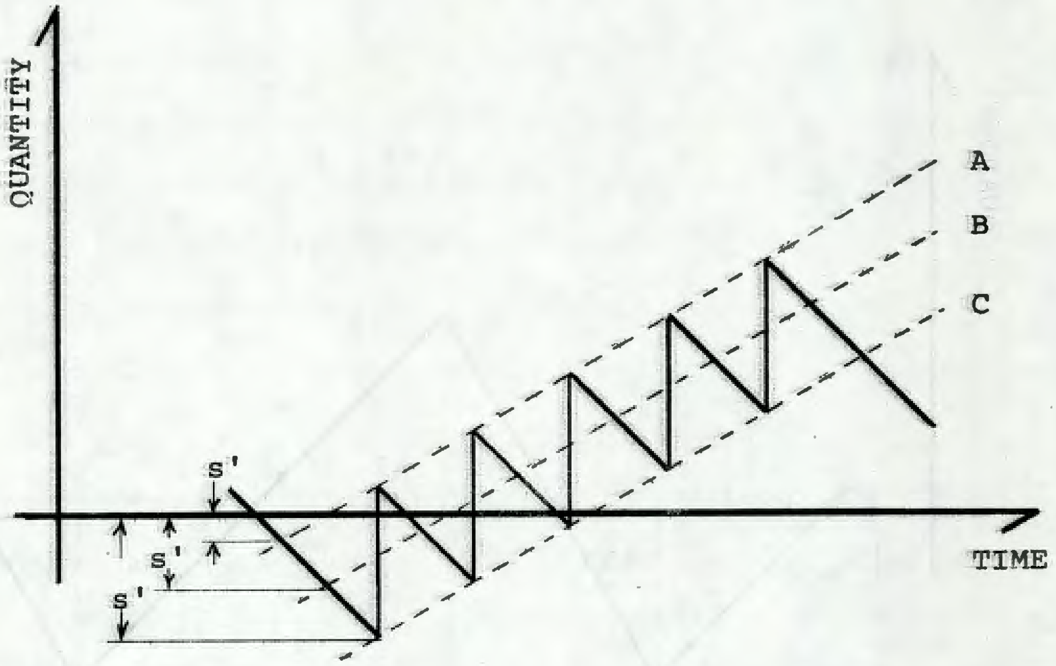


FIGURE 18

SHORTAGE MODEL

APPROXIMATION ~~THREE~~

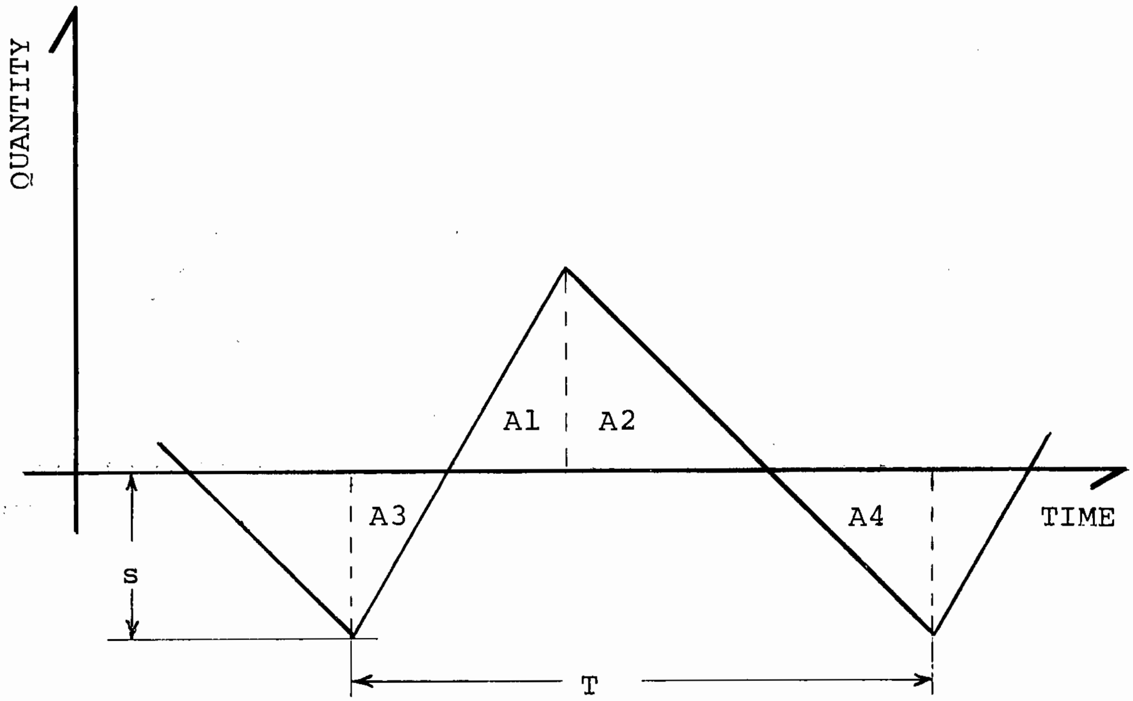


FIGURE 19

APPROXIMATE SHORTAGE MODEL

and s. Also recall $R = q/\tau$. Now in the approximating model consider the areas marked A1, A2, A3, and A4 on Figure 19. The holding cost per unit time is

$$\frac{(A1+A2)}{T}\, IC \; .$$

The shortage cost per unit time is

$$\frac{(A3+A4)}{T}\,\hat{\pi} + \frac{\pi s}{T} \; .$$

Therefore the applicable variable cost per unit time is

$$K = \frac{A\lambda}{nq} + \frac{B\lambda}{q} + \frac{IC}{2nq}\left[nq\left(\frac{R-\lambda}{R}\right) - s\right]^2\left(\frac{R}{R-\lambda}\right) + \frac{\hat{\pi}s^2}{2nq}\left(\frac{R}{R-\lambda}\right) + \frac{s\pi\lambda}{nq} \; .$$

Using the usual process we can determine n' and s', the optimal values for the approximating model, which are

$$n' = \frac{1}{q}\left[\frac{\hat{\pi}+IC}{\hat{\pi}}\left(\frac{R}{R-\lambda}\right)\frac{2A\lambda}{IC} - \frac{(\pi\lambda)^2}{\hat{\pi}IC}\right]^{\frac{1}{2}} ,$$

and

$$s' = \left(\frac{R-\lambda}{R}\right)\left(\frac{ICnq-\lambda\pi}{IC+\hat{\pi}}\right) .$$

Now we must decide which approximation to use. n* - n' for all three cases but in setting the reorder point we can use s* = s' + a where a is an adjustment to yield a more or less conservative policy. Increasing s* increases the reorder point and reduces the actual occurrence of shortages. a = 0 is equivalent to using the approximating line A of Figure 19 while letting $a = \lambda\tau$ is equivalent to the more conservative policy using the approximating line C of Figure 19. Any value of a can be used. The range

$0 \leq a \leq \lambda\tau$ corresponds to the more reasonable set of line approximations between line A and line C and the asymptotically exact value for a is $\lambda\tau/2$.

## 5.2 Probabilistic Parameters

The introduction of a probabilistic parameter introduces considerable complexity into any inventory model. This is particularly true of models which have realistic features, such as shortage costs, and are designed to be applicable to real inventory systems. The introduction of a demand distribution into a phased delivery model may be exceedingly difficult unless simplifying approximations are made. It is assumed that the reader who has gotten this far into this paper, is well acquainted with the basic single delivery probabilistic models for continuous review and periodic review. For the reader who is not, it will probably be necessary to study these basic probabilistic models in an inventory text such as Arrow, Karlin and Scarf [2] or Hadley and Whitin [1]. The following will be basically a comparative discussion and consider the modifications required rather than derive probabilistic models.

The modifications required are fairly easily divided into three groups. Incremental deliveries effect lead time, on hand levels and the frequency with which a risk of shortage is taken. Essentially,the desireable effects are based on the reduction of leadtime and stockage levels, while the deleterious effects arise from a more frequent risk of shortage. The three functions are taken up in order of increasing complexity.

It is quite probable that phased delivery models will result in earlier deliveries of the first units ordered. Unless constrained externally to the model, $q^*$ will always be less than the $Q^*$ which would result from a single delivery. This will usually mean the delivery quantity $q$ will be available for shipment a shorter time after the order is placed than will be $Q^*$. We have defined $\tau_0$ as the delay prior to receipt of the first delivery of material. Since safety stock is monotonically nondecreasing with respect to $\tau_0$, we can expect a reduced safety level (and therefore reduced cost) due to incremental deliveries. If the safety level is large enough to make the possibility of shortages after arrival of the first increment remote, then simply using $\tau_0$ for the lead time will take care of this change. If the problem of shortages later in the cycle is significant, then it would be difficult to get exact results but the distribution of demand over $\tau_0$, $\tau_0 + \tau_1$, $\tau_0 + \tau_1 + \tau_2$, and so on, would have to be computed and used in the probabilistic model.

Fortunately, the problem of determining the reduced level of on hand stock is simplified by the fact that the reduction in net inventory due to the incremental deliveries will be known precisely. The problem is that the reduction in holding cost is based on the reduction in average on hand stock and not on average net inventory. Once again, it is clear that if shortages are rare and it is extremely improbable that shortages will occur after the first lot is received, there is no problem. On hand and net inventory levels will be identical over the period of time for which the deliveries are placed. This means we can simply subtract from the probabilistic computation of holding cost,the fixed saving due to incremental deliveries.

If the probability of shortages after the first lot is received is significant, it will be necessary to calculate the expected value of on hand separately for each time period. That is, integrate over the time between order and first delivery, and between each pair of deliveries separately, using the final distribution of inventory position in one interval as the starting distribution of inventory position for the next interval. This is not particularly tractable even when the distribution assumed is simple. However, as soon as it becomes obvious that the probability of shortages is insignificant, the technique in the previous paragraph can be used for the remaining part of the cycle. The increased frequency of shortages is a very important consideration when $q_1$ is only a little larger than $\lambda\tau_1$, or, more generally, when the sum of the quantity received in the first few deliveries is not much larger than the demand during that period. When $q_1 >> \lambda\tau$, and receipts occur more rapidly than demand is likely to be generated, shortages will probably occur only once per cycle. However, there will always be some more risk of shortage in an incremental delivery model than in a single delivery model because the build up of assets is phased and not instantaneous.

When special conditions exist, there may be fairly simple ways of handling shortages. For example, suppose we have r units on hand at the time an order for n increments of size q each, at intervals $\tau_i$ as defined earlier is placed. Then the expected number of shortages given the distribution of demand over a time period t is f(x,t), is given by the following sum of integrals.

$$\int_{a^0=r+1}^{\infty} (x-a^0) f(x,\tau_0)\,dx + \int_{a'=r+1+q}^{\infty} (x-a^1) f(x,\tau_0+\tau_1)\,dx + \ldots$$

$$+ \int_{a^i=r+1+iq}^{\infty} (x-a^i) f(x, \sum_{j=0}^{i} \tau_i)\,dx + \ldots$$

In general, it will always be possible to consider the problem in n+1 time intervals; from the placing of an order to delivery 1, delivery 1 to delivery 2, ... , delivery n - 1 to delivery n, and delivery n to next order. The first n intervals correspond to $\tau_i$ $i = 0, \ldots, n - 1$. Define $\tau_n$ as the interval between the last (nth) delivery and the next order placement.

In this model, when lead time is stochastic, only the first interval varies in length, since we assume the $\tau_i$ are fixed for all $i = 1, \ldots, n - 1$. This introduces a real problem, since the distribution of lead time demand is based on both the distribution of lead time and of demand. The lead times for all deliveries except the first, have the form of the sum of two random variables; one the lead time demand for $\tau_0$ and the second, the demand over a fixed time period. However, we presume that if $\tau_0$ is large relative to the $\tau_i$ we can use the lead time demand distribution for all deliveries simply by replacing $\tau_0$ with the appropriate sum of $\tau_i$. If $\tau_0$ is small relative to the $\tau_i$ we can use a simple demand distribution for the various fixed intervals using the expected value of $\tau_i$.

With this assumption, we can compute the desired holding cost and shortage cost terms by summing the results for the holding and shortage cost terms for each of the n + 1 intervals computed as follows. For the jth interval j = 0, 1, 2, ..., n use a constructed reorder point of

$$r_j = \sum_{i=0}^{j} q_i ,$$

where $q_0 = r$, the conventional reorder point. Use

$$\sum_{i=0}^{j} \tau_i$$

as the parameter for mean lead time. For limits on the jth time interval integration use

$$\sum_{i=0}^{j-1} \tau_i \quad \text{and} \quad \sum_{i=0}^{j} \tau_i ,$$

as lower and upper limits respectively where

$$\sum_{i=0}^{-1} \tau_i \equiv 0 .$$

This will usually be a tedious and lengthy computation, but each integral in the sum will be tractable if the corresponding single delivery model would be tractable. If not, expected value models would have to serve.

## 5.3 Reducible Models

Whenever we can delay determining the size of the ith lot beyond the time the contract is placed, we can reduce the model into stages. The total order quantity is determined by one model. The optimal shipment lot sizes can be determined in stages, using a second model at the time the decision must be made, say one month prior to the delivery. This is a typical contractual situation. Often annual contracts are placed for an approximate quantity and calls are placed each month for the quantity needed. There is a potential problem in determining the duration of the contract but generally this can be ignored since it will be set by policy. The real problems are in determining the optimal interval between orders and the optimal quantity to be ordered. The optimal interval between placement of call orders against the contract can be found using a modified version of the model used to determine the optimal cycle length when operating on a periodic review basis. Having determined the optimal interval, the optimal quantity and safety level can be calculated using a conventional periodic review model with the incremental lot order cost used in place of the contract placement order cost. These problems are well covered in the literature.

In a forthcoming paper sponsored by the U. S. Army Material Command, Deemer and Hockstra [3] have developed a model of this type in detail. They call it a decoupled model since the costs can be decoupled into those dependent upon the contracting cycle and those dependent upon the safety level and the two problems separated. They assume a fixed delivery cycle of one month. This model is useful if the control over each delivery quantity is extensive and the decisions can be made as time progresses rather than in advance. Because of the paper referenced, this problem will not be further discussed here.

# CHAPTER 6

## STOCHASTIC DELIVERY QUANTITIES

Delivery quantity may be stochastic for a fairly extensive class of items, characterized by the fact that the cost may be significantly lower if the source is allowed some latitude in the quantity to be provided. One example of this class would be a valve machined from an expensive casting, which may have flaws which are first discovered during machining. It is frequently necessary for productive efficiency to cast all of the blanks at one time, in advance. There will then be a random number of blanks found to be flawed during machining. If the contract is for an exact quantity, the problem of how many blanks to cast is an interesting problem itself. It is costly to have too few perfect blanks and have to cast some more, losing both time and set up costs. It is also costly to cast too many blanks for disposition as surplus. Since these costs will be passed on to the purchaser, it will often be cheaper for the inventory manager to negotiate a contract to accept some range of quantity (say 400 ± 10%) at a fixed cost per unit. This will result in procurements for which the delivery quantity is stochastic. The delivery quantity need not even have an expected value equal to the quantity ordered.

For example, if the cost of producing surplus blanks is very small, (or their salvage value is high) relative to the cost of running short of blanks, it will be desireable to produce many blanks and the expected delivery quantity will be close to the upper end of the range. On the other hand, if the blanks are expensive (make up a large proportion of the final cost of the item) and have little or no salvage value, then relatively few blanks will be produced and the expected quantity delivered will be in the lower part of the allowed range. The reader will note that the producer's problem may be treated as a modified newsboy problem.

Suppose one has determined an optimal operating policy based on a model which assumes delivery of the exact quantity ordered. When placing the contract, the price quoted for an exact number of units is $100 each, but the contractor offers a price of $98 each if you will accept a variation of plus or minus 10% in the delivery quantity. He is willing to do this for reasons similar to those mentioned earlier. Is our computed policy optimal, or even near optimal, now that we have this additional option? To find out we will investigate the revised optimal policy.

We know that the cost function K is relatively flat in the vicinity of its minimum, so a change of plus or minus 10% in the receipt quantity will probably not increase costs sufficiently to offset the reduced unit cost. The ratio of the cost of a nonoptimal Q to the cost of Q* is given by

$$\frac{1}{2}\left(\frac{Q}{Q^*} + \frac{Q^*}{Q}\right) .$$

For a nonoptimal Q, which is off by a factor of 2 in either direction, there is an increase of approximately 25% in the cost. For a nonoptimal Q, which is within plus or minus 10% of the optimal value, this ratio is less than 1.005. This implies an increase in operating cost of less than $\frac{1}{2}$ of 1% compared to a 2% discount on unit cost. If the annual operating cost is less than 4 times the value of annual demand, it will pay to accept the offer of the variable quantity at the lower cost per unit. However, there will be some increase in operating cost, due to the uncertainty about the quantity that will actually be delivered, which has not yet been considered.

Let us divide the discussion into two parts. We will consider the increased cost due to uncertainty later. First, assume that the lead time demand is always less than the smallest possible delivery quantity. Then we will have no orders outstanding to contribute uncertainty at the time an order is placed, so that the reorder point computation can be made as if there were no variation in Q. However, by adjusting the quantity, we may be able to further decrease costs. Assume that the density function f(Q) for the delivery quantity Q based on an order of size Q; is known. We are actually supplying one of the parameters of the distribution f(Q) when we determine Q'. Now, we can choose Q' so that the expected value of K is minimal. Let r be the smallest allowed ratio of Q to Q' and r' the largest. We want to find Q' such that

$$\int_{rQ'}^{r'Q'} K(Q)f(Q)dQ$$

is a minimum.

Suppose, for example, that

$K(Q) = a/Q + bQ$, (roughly the form of $K_w$),

and that $Q$ is distributed uniformly from $.9Q'$ to $1.1\,Q'$ (±10%). We want to choose $Q'$ so that the expected value of $K$ will be a minimum. That is, minimize the following integral with respect to $Q'$:

$$K' = \int_{.9Q'}^{1.1Q'} \frac{a}{Q} + bQ \quad \frac{1}{.2Q'} dQ$$

$$K(Q') = \frac{a'}{Q'} + b'Q'$$

where $a' = 5a \ln(11/9) \doteq 1.0024\, a$, and $b' = b$. The optimal value for $Q'$ is very close to the usual optimal $Q$, but is slightly larger since $a' = a + \varepsilon$. In this example, $Q' = 1.0012Q_w$. To interpret this result, recall that we are considering a uniform weighting of $K$ at the various values of $Q$. $K$ is relatively flat in the vicinity of its minimum, but increases somewhat more rapidly as $Q$ decreases, than it does when $Q$ increases, so we prefer to be "off optimal" on the high side.

As a second example, suppose the contractor will usually produce and ship a quantity near the upper end of the allowed range. This is quite likely in real systems, since the manufacturer will generally "aim" for a quantity in the upper end of the range to maximize the sales profit.

Also, if the quantity actually produced is too small, he will make more, while if the quantity produced is too large, he supplies the maximum quantity allowed by the contract. This has a tendency to cause the delivery quantity distribution to be skewed to the right, with positive point probability at its right end point, even if the underlying production distribution is symmetric. For simplicity, assume a triangular distribution:

$$f(Q) = \frac{50(Q-.9Q')}{(Q')^2} \quad , \; f(.9Q') = 0 \; , \; f(1.1Q') = \frac{10}{Q'} .$$

Now our problem is to minimize the expected value of K with respect to Q'. That is minimize

$$K = \int_{.9Q'}^{1.1Q'} (a/Q+bQ) \, \frac{50(Q-.9Q')}{(Q')^2} dQ$$

$$= \frac{a'}{Q'} + b'Q'$$

where $a' = 10a - 45a \ln(1.22\overline{2}) \doteq .98a$, and $b' = 10.035b - 9.0b = 1.035b$. This implies we should choose a smaller value for Q' than $Q_W$. Q' is approximately $.975Q_W$. Intuitively, this is fairly easy to understand. We "expect" to get more than we order, so we "order short" to get the quantity we "really want". Note, however, that the changes above will be fairly nominal. The difference in the values of Q' and $Q_W$ is very small and the flatness of the cost function means that the cost saving is likely to be small so that it does not offset the extra cost of computation. The savings due to ordering Q' instead of $Q_W$ in these two examples are .02 and .03 percent respectively.

We are now ready to consider the more difficult problem of the cost of uncertainty. Suppose the lead time is long relative to the time between orders, so that several shipments are outstanding when we place an order. These orders will be received during lead time. Normally, we would know the total asset picture exactly, but we now have an order quantity which is stochastic. In calculating the optimal reorder point, we need to consider the difference between two random variables, instead of the difference between a constant and a random variable. The problem may be further complicated by the fact that one of these random variables is actually a sum of random variables which may not be independent. Assume for a moment, however, that they are independent and define Y as the total quantity to be received during a lead time. The expected value of Y is n times the expected value of Q and it is relatively simple in theory to calculate the density of Y, as a convolution, and then from the density of X and this density of Y to calculate the density of $Z = X - Y$. We can use this density of Z in place of the density of demand X in the conventional models. Techniques for finding the densities of Y and Z can be found in Parzen [4].

However, it will not generally be possible to work with this sort of density, since the resulting function will be a special density and not a simple or tabulated one. Also it is not likely in practical systems that the basic starting distributions or their parameters will be known.

If we were fortunate enough to have adequate sample data, we could estimate a distribution family and parameters for the net inventory loss (or gain) during a lead time. Seldom will we have adequate data to do this, so it is usually necessary to assume a distribution, which seems appropriate, and estimate its parameters from the estimates of the parameters of f(x) and f(Q).

It is reasonably clear that a fairly good estimate of the mean of net lead time demand will be the expected value of lead time demand minus the expected value of receipts (not n times the quantity ordered). Use of the parameter $m = m_X - nm_Q$ as the mean appears most reasonable. It is equally obvious that the variance $\sigma^2$ should be taken as greater than $\sigma_X{}^2$. The question of how much greater is a problem, however. If we were convinced that the demand quantity during lead time and the delivery quantities $Q_1$, $Q_2$, ..., $Q_n$ were each independent of the other, then $\sigma^2 = \sigma_X{}^2 + n\sigma_Q{}^2$ would be appropriate. If we were convinced that the $Q_i$ were completely dependent, that is $Q_i = A$ for all i = 1, ... n, then $\sigma^2 = \sigma_X{}^2 + n^2\sigma_Q{}^2$. On the other hand, the variance should be less than $\sigma_X{}^2 + n\sigma_Q{}^2$ if there is a tendency for things to even out. That is, if successive orders are negatively correlated, which may occur when the producer is trying to make corrections to his policy, the variance of total receipt quantity may be reduced below the variance assuming independent delivery lot sizes. However, it appears reasonable to assume net leadtime demand has mean $m = m_X - nm_Q$ and variance $\sigma^2 = \sigma_X{}^2 + n\sigma_Q{}^2$ unless we know the relationship between successive order sizes.

The significant factor is that the variance of lead time demand should be increased by at least some amount and this will result in a higher safety level and/or greater risk of shortage, either of which increases operating costs.

# CHAPTER 7

# OTHER RELAXATIONS OF THE CONVENTIONAL DELIVERY ASSUMPTIONS

## 7.1 Stochastic and Variable Production Rates

In the previous chapters it has been assumed that the production rate was either constant or a fixed function of time with no probabilistic element and that the inventory manager had no control over the rate of production. A production rate may be stochastic for several reasons. The production rate may be stochastic because it is a function of a stochastic process which generates imperfect output. Machinery breakdown, strikes, emergency orders and other exogenous factors also influence the production rate. For example, the production rate (delivery rate) of a valve might be influenced by any of the following: the availability of brass for casting, timely delivery of brass to the foundry, other high priority foundry projects, the actual percentage occurrence of flaws in the castings, variation in the quality of brass, foundry equipment, or personnel performance, shipment from foundry to machine shop, machine breakdowns, high priority shop projects, etc.

In this section we relax the assumptions in two ways. First, we will consider a stochastic production rate $\Psi$ with probability density function $f(\Psi)$, $\Psi \geq 0$. Consider a finite continuous production rate model for which both demand and production are stochastic. See Figure 20. Let $f(x,t)$ be the probability density function of the demand X during an interval of length t. Let $g(y,t)$ be the probability density function of the production Y during the interval t, derived from $f(\Psi)$. We assume all parameters of $f(y,t)$ and $q(y,t)$ are known. During the first part of a cycle, on hand stock builds up at the stochastic rate $Z = Y - X$. Once a total quantity of Q has been produced, production stops and the stock depletes at the rate X.

Let $h(z,t)$ be the density function of the net increase of inventory during the interval,t, defined as

$$h(z,t) = \int_{-\infty}^{\infty} g(z+x,t)\ f(x,t)\ dx = \int_{-\infty}^{\infty} g(y,t)\ f(y-z,t)\ dy\ .$$

We use the product of the individual densities as the joint density since it is reasonable to assume X and Y are independent. It is quite realistic to allow the value of the random variable Z to be negative. This can easily happen during some time interval when demand is unusually large and production is unusually small. However, it is necessary for feasibility for Y to be stochastically greater than X, so that the expected value of $Z = Y - X$ will be greater than zero.

We can use the distribution $h(z,t)$ to find the expected holding and shortage costs during the production phase of the cycle and the distribution $f(x,y)$ to find the expected holding and shortage costs during the depletion phase of the cycle. However, we still have a problem, since both

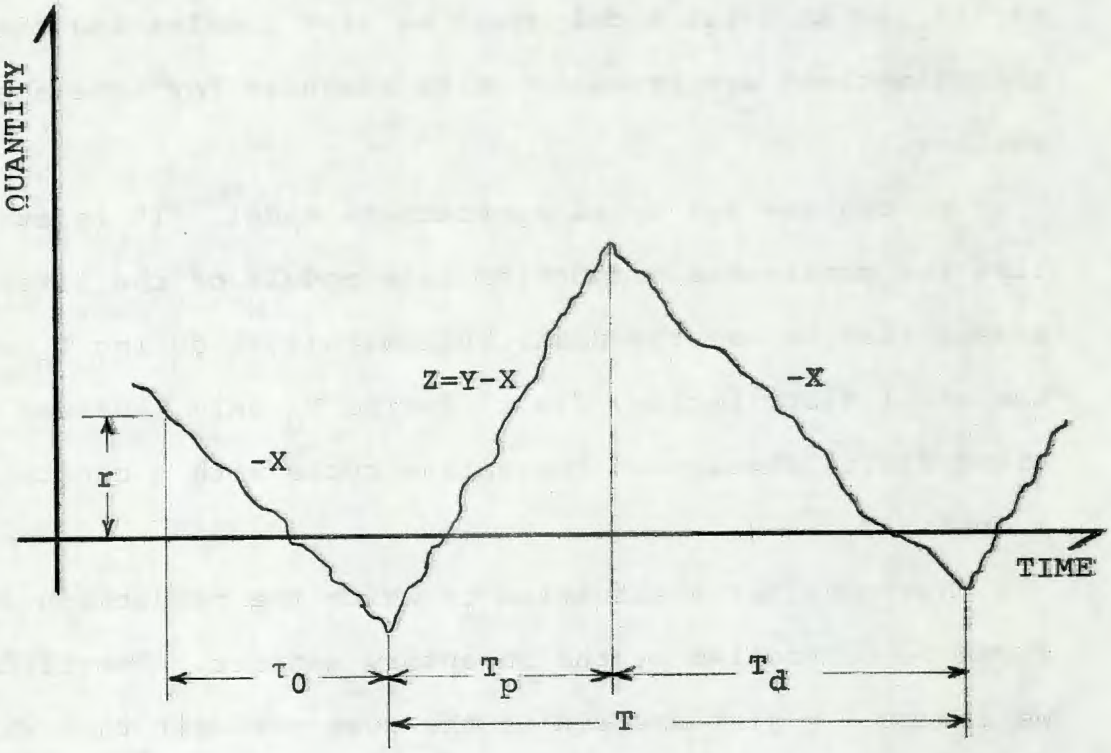


FIGURE 20

STOCHASTIC DEMAND AND PRODUCTION

the length of the production phase and the on hand quantity at the end of the production phase are stochastic. These values can not generally be found exactly, so we must use expected values for $T_p$, $T_d$, $\tau_0$, and on hand at the end of the production phase which is approximately $r - \lambda\tau_0 + (\Psi-\lambda)T_p$ . An exact model would be very complex and these approximations are probably quite adequate for some applications.

We can now set up an approximate model. It is exactly like the continuous production rate models of the literature, except that we use the distribution, h(z,t) during $T_p$ and the usual distribution, f(x,t) during $T_d$ only, instead of using f(x,t) throughout the entire cycle with a constant $\Psi$ during $T_p$ .

Now consider a situation in which the production rate $\Psi$ can be controlled by the inventory manager. Specifically, we assume $\Psi = g(x)$ where x is the cost per unit time which is passed on to the customer by the manufacturer. The cost of production as a function of the production rate is generally accepted by economists to be convex from below, which implies some finite optimal rate greater than zero. There is usually an economy of scale effect at the lower production rates. Then at rates above the optimum, the production rate can be increased by the addition of machinery, by subcontracting part of the work or by producing on overtime at premium wages. These costs can be expressed in terms of decreased or increased unit cost,but the effect of this cost is clearer when isolated. Although a quick reference to the finite production rate model of section 1.3 will suggest that a production rate of $\Psi = \lambda$ is optimal, this applies only if we are operating on a portion of the $\Psi$ range where the cost is near minimal. In general, to find an optimal we would add the cost $C(\Psi)$ to the variable cost so that

$$K = C(\Psi) + \frac{A\lambda}{Q} + \frac{ICQ}{2}\left(\frac{\Psi-\lambda}{\Psi}\right) .$$

This can be solved for optimal values of $\Psi$ and Q by the usual techniques. We also have a constraint to consider, since for feasibility, $\Psi^* \geq \lambda$. The same simple modification can be made to any finite production rate model.

There is also a meaningful single period model to be considered, when stock has dropped below the desired on hand level and we want to regain our steady state optimum. To do this, we wll have to increase $\Psi$. For simplicity and clarity, we consider the model shown in Figure 21. We have b backorders and a deterministic demand rate $\lambda$. The backorder cost is $\hat{\pi}$ per unit per unit time.

If we choose a $\Psi$ that is much larger than $\lambda$ we will save on the cost of backorders, but it will cost us more to achieve the rate $\Psi$. If we let $\Psi = \lambda + \varepsilon$ where $\varepsilon$ is small, then it will cost us little extra to get the faster production, but the backorders will last for some time and be expensive. Is there an optimal $\Psi$?

The costs of importance in determining the optimal value of $\Psi$ for this model are the costs of the backorders

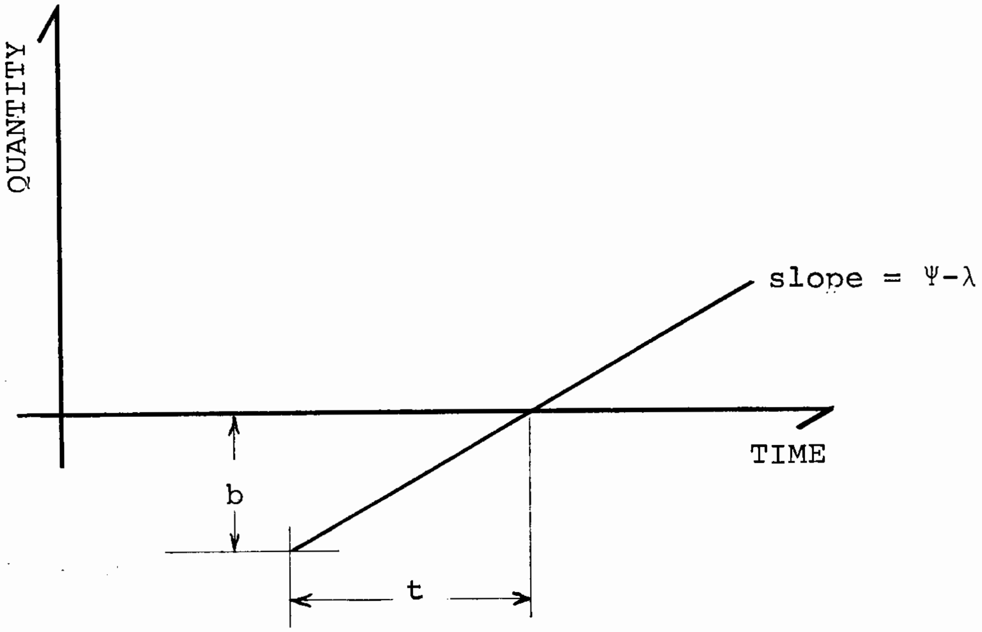


FIGURE 21

SINGLE PERIOD, VARIABLE Ψ MODEL

during the time t until we are back in steady state optimal operation and the cost of the accelerated $\Psi$ during the same period. Let cost = $C(\Psi)$ be the cost of the rate $\Psi$ (the inverse function of $\Psi = g(x)$.

Now the total cost to be minimized with respect to $\Psi$ is

$$K = C(\Psi)t + \frac{\hat{\pi}}{2} bt = \frac{C(\Psi)b}{(\Psi-\lambda)} + \frac{\hat{\pi}\, b^2}{2(\Psi-\lambda)} .$$

For analytic convenience, we may make a change of variable to $t = \frac{b}{(\Psi-\lambda)}$ and find the optimal value of t. If we are producing at or above the optimal rate $\Psi_0$, then, in this range, $C(\Psi)$ is monotonically increasing in $\Psi$. Typically, for some range up to $\Psi$ max, we can increase our rate by some fixed amount for each hour of overtime. This results in a linear function. Suppose $C(\Psi) = a + a'\Psi$. Then

$$C(\lambda+\frac{b}{t}) = a + a'\lambda + \frac{a'b}{t} ,$$

and

$$K = (a+a'\lambda+\frac{a'b}{t})t + \frac{\hat{\pi}}{2} bt .$$

Now

$$\frac{\partial K}{\partial t} = a + a'\lambda + \frac{\hat{\pi}}{2} b = 0 ,$$

which is not a function of t and implies $t^* = 0$ or $t^* = \infty$.

$$K = a'b + t(a+a'\lambda+\frac{\hat{\pi}}{2}b)$$

is monotonically increasing with t. Therefore $t^* = 0$ and $\Psi^* = \lambda + \frac{b}{t^*} = \infty$. We interpret this to mean produce at $\Psi$ max.

This is intuitively reasonable, since the linear cost of increasing production will imply a fixed cost of making up the shortage regardless of the time taken, and the faster the shortage is made up the lower the backorder cost.

Now suppose $C(\Psi) = a\Psi^2$. This is a realistic approximation of the cost curve above $\Psi_0$, the optimum production rate. Now

$$K = a\Psi^2 t + \frac{\hat{\pi}}{2} bt.$$

Let us substitute $\frac{b}{\Psi-\lambda}$ for t and solve for $\Psi^*$ directly.

$$K = \frac{ab\Psi^2}{\Psi-\lambda} + \frac{\hat{\pi} b^2}{2(\Psi-\lambda)},$$

and

$$\frac{\partial K}{\partial \Psi} = ab\left[\frac{2\lambda}{\Psi-\lambda} - \frac{\Psi^2}{(\Psi-\lambda)^2}\right] - \frac{\hat{\pi} b^2}{2(\Psi-\lambda)^2} = 0$$

implies

$$\Psi^* = \lambda + \sqrt{\lambda^2 + \frac{\hat{\pi} b}{2a}}.$$

Similar results can be obtained for any function $C(\Psi)$.

## 7.2 Variable Lead Times and Expediting

The advantages of shortened lead time are not apparent in deterministic models since there is no uncertainty involved. The major advantage of shortened lead time is due to the reduction in variance of lead time demand and is apparent in probabilistic models. However, in general, a reduced lead time has other benefits which are overlooked in most analytic models since the assumptions made exclude these effects. Reduced lead time reduces the risk of becoming overstocked and reduces the reaction time in situations where parameters are changing or subject to change, and when stock is lost or destroyed. Reduced lead time has particular value when trends exist,or when sudden changes in demand rate or costs occur, or when the system is in trouble.

The discussion of lead time control is divided into two parts. We will first consider routine reduction of lead time as a continuing practice in a settled steady state system with stochastic demand. This will be formulated as a trade off between the cost of decreasing lead time and the expected cost saving due to decreasing safety level requirements. The second lead time reduction considered will be referred to as expediting and differs from routine lead time reduction,since we will assume that an improbable event has occurred during lead time and we face unusually large shortages prior to the planned delivery. This is formulated as a single period model. In real systems the first model would be part of the routine stock replenishment computations.

The second model is applicable only when the system has gotten into temporary trouble due to unusually large demand, or loss of stock,or when making initial procurements on new items. To formulate a routine lead time optimization model we can modify any of the convention models in the literature by adding the decision variable t. We assume that a normal lead time, $\tau_N$, is available at no extra cost and that the cost of reducing lead time by t time units is c(t). We then replace the fixed value for lead time throughout the model with $\tau_N - t$ and add the cost term c(t). This results in a modified model to be optimized with respect to the original decision variables and t.

For example, consider a very simple approximate model as shown in Figure 22. The objective is to minimize the cost of holding an inventory which is adequate to insure that the probability of one or more shortages in a cycle is less than P. If the normal lead time, $\tau_N$, and the lead time demand distribution are known, we can find an optimal t as follows. The holding cost is approximately $\left[\frac{Q^*}{2} + b\right] IC$ where b is the buffer stock required to keep the probability of shortage down to P. The cost of reducing lead time is c(t). Recall b will be a function of lead time, lead time demand and P. The reorder point, r, will be equal to b plus the expected lead time demand, $\lambda\tau$. The variable cost dependent upon t is,

$$K = c(t) + ICb(t).$$

Suppose for example that $c(t) = 200t^2$, the lead time demand is normal with mean $\lambda t$ and variance $t^2$, $\lambda = 20$, $IC = 40$, $P = .05$ and $\tau_N = .5$.

Now we know that to insure we have at most a .05 chance of shortage during lead time when the demand is normally distributed, we must have stock on hand at the time of order of at least the mean lead time demand plus 1.65 standard deviations. This means that b(t) is equal to 1.65

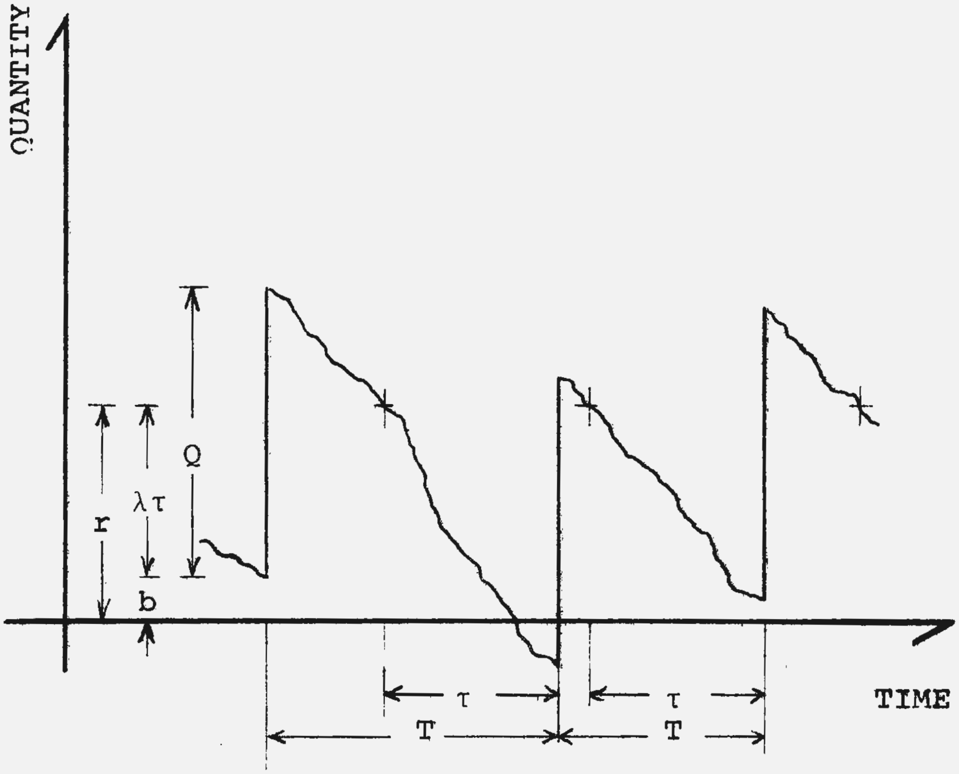


FIGURE 22

OPTIMAL LEAD TIME MODEL

times the standard deviation $\tau_N - t$. We can now write

$$K = 200t^2 + 40(1.65)(.5-t).$$

Solving for the t which yields minimal costs, we find,

$$t^* = .165.$$

A single period model for expediting would be a fairly simple modification of the model above, but we would use the actual value of inventory position instead of the reorder point in the calculations. Allen and D'Esopo [5] present an analysis where an option exists to expedite delivery at a fixed extra cost. They determine an optimal policy for making a decision whether to expedite, but the analysis is limited by its assumptions. Generally, optimal inventory policies minimize long term expected cost so there will often be specific points in time where improbable events have occurred and remedial action such as expediting will result in additional savings. The frequency with which such situations arise in constrained budget systems, such as the military inventory systems, is quite sufficient to warrant further investigation.

## 7.3 Multiple Source With Different Delivery Patterns

Generally, an inventory manager has several sources for stock replenishment. It will be purely accidental if all sources offer the same delivery conditions. Some work has been done in considering multiple source procurement, notably the paper by W. J. Fabrycky and Jerry Banks [6] which considers a multiple source multiple item inventory problem. Present practices usually choose the procurement source solely on the basis of lowest item cost. It is obvious from the considerations of the previous chapters that delivery conditions can influence total cost in a significant fashion. It may be cheaper over all to pay more for the items to get favorable delivery, particularly when the demand has a high variance to mean ratio or when the system is in short supply. Unfortunately, purchase rules generally preclude applying such enlightened management procedures. Dramatic changes in bid evaluation ground rules would be required. No attempt is made here to consider this facet of the inventory problem in detail. Several of the previous models can be applied in appropriate situations. At worst, it may be feasible to compute and compare the minimum cost per unit time for each available source of supply, and choose the source associated with the lowest of these minimums.

Some comments on the Fabrycky-Banks paper are included here in order to provide an interface between the problem they approach and the problem of delivery variation. They consider the problem of multiple items with multiple sources and give a dynamic programming technique for solution of the deterministic demand and lead time case. Their paper has an obvious application to the problem of relaxing delivery assumptions, although for each "source" the conventional delivery assumptions are made. A separate order cost, unit price, leadtime and delivery rate are assigned to each "source". "Source" is placed in quotation marks since "source" in their formulation can as well apply to various modes of delivery from a single manufacturer, and provides a method both for choosing between distinct sources and between delivery options from a single source. It is directly applicable to choosing optimally between specific alternative delivery schedules for which various lead times and delivery rates are available at different order costs and unit prices.

Continuous models can be fit into their dynamic programming model by investigating a set of fixed options and then reinvestigating another set of more finely spaced options in the vicinity of the minimum cost parameters. This procedure can be continued until the desired degree of accuracy is obtained.

# CHAPTER 8

## SUMMARY AND CONCLUSIONS

There is much work to be done in the area of relaxed delivery assumptions. Unfortunately, it does not seem likely that any general cohesive models will be forthcoming in the near future. Unified treatment models become exceedingly complex and intractable. The only feasible approach at present appears to be to generalize delivery assumptions by setting up alternative models with different assumptions about the delivery conditions.

Applicable models with more than one stochastic parameter are always difficult to handle. Exact models can be formulated only when very restrictive assumptions are made about the distributions of the random variables. Approximate solutions to models with simplifying assumptions made as required for algebraic and statistical tractability often end up far afield. The major problem is that the assumptions are dictated by mathematical convenience and not by the nature of the model. It may be much more difficult to assess the impact of such assumptions than to assess the impact of assumptions which are made about the model prior to any attempt at solution.

Approximate models which are deterministic or incorporate only simple probabilistic considerations may provide a manager with operation policies which are far superior to the techniques previously used.

The basic ideas approached in this paper were that deliveries may be phased over time, that receipt (production) may be a stochastic process as well as issue (demand), that the manager may have control over certain parameters usually taken as given ($\Psi,\tau$), and that the minimal cost policy may depend on the delivery conditions and options as well as the conditions usually taken into account.

There are several tentative conclusions that can be made based on the models and ideas in this paper. The conclusions drawn are:

1. Phased delivery will usually result in larger order quantities and lower cost.
2. The greater the inventory manager's control over the delivery conditions the cheaper a system can be run.
3. Shortening lead time has a direct economic value in stochastic systems and is often worth paying for.
4. When the manager has the option of determining the size of deliveries as the contract progresses, the model will reduce to a standard delivery model with an adjusted cost of order.
5. Stochastic order quantities are unlikely to effect the optimal order policy significantly unless the distribution is very skewed or unless lead time is much larger than cycle time.

6. Unless constrained, the optimal contract period will tend to be very large for most real systems. It will usually be a good approximation to place a contract for the longest period which common sense and the exogenous constraints allow.

7. It is often possible to calculate the optimal shipment size within a small error by using a modified purchase cost in the standard infinite delivery rate models.

8. In terms of the defined parameters, optimal shipment lot size is an increasing function of B, the lot order cost, and is generally independent of A, the contract placement cost, while contract duration is primarily a function of A and is relatively uninfluenced by B.

9. When delivery conditions vary among supplier's bids, delivery conditions should be considered in choosing the "low-bid". The smallest unit price may not indicate the most economical source.

10. Fractional delivery models, where fixed percentages of the ordered quantity are delivered at fixed intervals, do not result in a change from the one lot delivery order quantity, if shortages are not allowed or are exceedingly rare.

11. In probabilistic phased delivery models each delivery lot should be slightly larger than the demand expected prior to the next delivery since

the increasing variance due to stochastic demand (and delivery) will require deeper safety stock to maintain comparable protection against stock out and achieve minimum cost.

The subject for this paper was suggested by the author's experience in inventory management in the U. S. Navy Supply system. The Armed Services have become more cost conscious in recent years, and are among the leaders in computerized scientific inventory management. The scale of operations and the dollar volume makes even small percentage savings significant. There is a broad range of item characteristics represented and a wide selection of delivery conditions. There is an obvious need for extension of the conventional delivery assumptions to take advantage of the unconventional delivery opportunities available. Of particular interest is the desirability of annual contracts calling for phased delivery to keep the contracting cost down without raising on hand inventory levels.

This paper represents a first assault on the problem. It is clear that the models and techniques proposed have considerable limitations. However, they provide a starting point from which to proceed. It is also hoped that some provide a means to determine whether the anticipated savings will justify the expense of detailed analysis.

# BIBLIOGRAPHY

1. Hadley, G., and T. M. Whitin. Analysis of Inventory Systems, Prentice-Hall, Inc., 1963.

2. Arrow, Kenneth J., Samuel Karlin, and Herbert Scarf. Studies in the Mathematical Theory of Inventory and Production, Stanford University Press, 1958.

3. Deemer, Robert L., Djoerd Hoekstra. "Improvement of MIT Non-Reparables Model"; Inventory Research Office, U.S. Army Logistics Management Center, Frankford Arsenal, Philadelphia, Pennsylvania; a draft paper to be published as a U.S. Army Material Command Study.

4. Parzen, Emanuel. Modern Probability Theory and Its Applications, John Wiley & Sons, Inc., 1960.

5. Allen, Stephen G., and D. A. D'Esopo. "An Ordering Policy for Stock Items When Delivery Can be Expedited"; a paper presented at the 31st National Meeting of the Operations Research Society of America, New York City, May 31, 1967.

6. Fabrycky, W. J., and Jerry Banks. "A Heirarchy of Deterministic Procurement Inventory Systems". Journal of the Operations Research Society of America, Vol. 14, No. 5, September-October, 1966.

# INITIAL DISTRIBUTION LIST

| | | No. Copies |
|---|---|---|
| 1. | Defense Documentation Center<br>Cameron Station<br>Alexandria, Virginia 22314 | 20 |
| 2. | Library<br>Naval Postgraduate School<br>Monterey, California 93940 | 2 |
| 3. | Director, Systems Analysis Division<br>Office of the Chief of Naval Operations<br>Washington, D. C. 20350 | 1 |
| 4. | Lt. J. M. Cook, S. C., USN<br>(Code 045)<br>Naval Supply Systems Command<br>Washington, D. C. 20360 | 1 |
| 5. | Lt. Cmdr. Paul A. Dollard, S. C., USN<br>Fleet Material Support Office<br>(Code 97)<br>Naval Supply Depot<br>Machanicsburg, Pennsylvania 17055 | 1 |
| 6. | Captain N. R. Harbaugh, S. C., USN<br>Director Research and Development Division<br>Naval Supply Systems Command<br>Washington, D. C. 20360 | 2 |
| 7. | Mr. James Prichard<br>(Code 04E)<br>Naval Supply Systems Command<br>Washington, D. C. 20360 | 1 |
| 8. | Mr. Bernard B. Rosenman<br>Chief, AMC Inventory Research Office<br>U. S. Army Logistics Management Center<br>Frankford Arsenal<br>Philadelphia, Pennsylvania 19137 | 1 |
| 9. | Prof. David A. Schrady<br>Department of Operations Analysis<br>Naval Postgraduate School<br>Monterey, California 93940 | 5 |

10. Prof. Peter W. Zehna 1
Department of Operations Analysis
Naval Postgraduate School
Monterey, California 93940

11. Library, Operations Analysis Department 1
Naval Postgraduate School
Monterey, California 93940

12. Supplemental Distribution 75
Deliver to Operations Analysis Secretarial Office
Herrmann Hall, Room W520
Naval Postgraduate School
Monterey, California 93940

## DOCUMENT CONTROL DATA - R & D

*(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

| | |
|---|---|
| 1. ORIGINATING ACTIVITY *(Corporate author)*<br>Naval Postgraduate School<br>Monterey, California 93940 | 2a. REPORT SECURITY CLASSIFICATION<br>Unclassified<br>2b. GROUP |

3. REPORT TITLE

OPTIMAL INVENTORY OPERATING POLICIES UNDER RELAXED DELIVERY ASSUMPTIONS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Thesis

5. AUTHOR(S) *(First name, middle initial, last name)*

COOK, John Morse, Lieutenant, Supply Corps, USN

| | | |
|---|---|---|
| 6. REPORT DATE<br>June 1968 | 7a. TOTAL NO. OF PAGES<br>125 | 7b. NO. OF REFS<br>6 |

| | |
|---|---|
| 8a. CONTRACT OR GRANT NO. | 9a. ORIGINATOR'S REPORT NUMBER(S) |
| b. PROJECT NO. | |
| c. | 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* |
| d. | |

10. DISTRIBUTION STATEMENT [redacted]

| | |
|---|---|
| 11. SUPPLEMENTARY NOTES | 12. SPONSORING MILITARY ACTIVITY<br>Naval Postgraduate School<br>Monterey, California |

13. ABSTRACT

Historically, the inventory control problem has been analyzed under the assumption that the exact quantity ordered was delivered in one lot, at the end of a probabilistic or deterministic lead time, or, alternatively, that receipts are the direct result of a continuous, fixed rate production process. In this paper, the effect of relaxation of these assumptions is considered. It is quite common in large inventory systems to have the total quantity ordered delivered in increments phased over time. Contracts providing for delivery of short or excess quantities within tolerances are also frequently encountered. Production rates may be stochastic due to work stoppages, the diversion of resources to meet higher priority requirements and other external factors. A family of models, with a variety of delivery conditions, are presented. The optimal operating policies and costs of these models are compared to the policies and costs resulting from models with conventional delivery assumptions.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

A-31408

| 14 KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| INVENTORY CONTROL | | | | | | |
| PHASED DELIVERY | | | | | | |
| RECEIPT | | | | | | |

[redacted]

[redacted]

thesC74724
Optimal inventory operating polici
DUDLEY KNOX LIBRARY
3 2768 00421982 4
DUDLEY KNOX LIBRARY